

# Teaching Logic for Computer Science: Are We Teaching the Wrong Narrative?

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## Abstract

In this paper I discuss what, according to my long experience, every computer scientist should know from logic. We concentrate on issues of modeling, interpretability and levels of abstraction. We discuss what the minimal toolbox of logic tools should look like for a computer scientist who is involved in designing and analyzing reliable systems. We shall conclude that many classical topics dear to logicians are less important than usually presented, and that less-known ideas from logic may be more useful for the working computer scientist.

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*Even in teaching mathematics we can at least attempt to teach the students the flavor of freedom and critical thought, and to get them used to the idea of being treated as humans empowered with the ability of understanding.*

Roger Godement, Cours d'Algèbre, Hermann, Paris 1966

## 1 Introduction

In the last few years we see contradictory developments concerning teaching logic for Computer Science undergraduates. On the one side, the importance of logical tools in hardware and software engineering, artificial intelligence, and database management, including the new trend of handling large data, is widely recognized. The Vienna Summer of Logic of 2014<sup>1</sup> attracted over 2500 researchers, but the traditional central logic conferences (ASL Logic Colloquium, LICS and CSL) played only minor part in terms of attendance. Most attendees were practitioners of logic, benefiting from and contributing to the the *unusual effectiveness of logic in computer science*, [9]. On the other side, leading Computer Science Departments<sup>2</sup> have taken logic courses off their compulsory curriculum of the 3 year program. The argument behind such a decision does not question the importance of logic, but cites the scarcity of available time for teaching the essentials.

The discrepancy between the success of logic in Computer Science and its relegation from the Computer Science curricula may indicate that the standard logic courses are outdated and miss their point. Teachers of logic in Computer Science are often still teaching

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<sup>1</sup> Vienna Summer of Logic 2014 <http://vsl2014.at/>

<sup>2</sup> ETHZ Zurich, is a good example. There was strong support to keep logic in the curriculum, but finally it was dropped, after the time and priorities argument won the vote by a small margin

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courses which are a mix of formalizing logical reasoning, meta-mathematics, and the fading reverberations of the famous crisis of the foundations of mathematics. By doing so, they are contributing to the disappearance of their courses from mainstream undergraduate education in Computer Science. We have to rethink which aspects of logic matter for the Computer Science undergraduate programs. I want to argue, that the fascination with completeness and incompleteness theorems, the beauty of the compactness theorem and its applications in infinite model theory, and the focus on first order logic in general, are didactically counterproductive.

I was trained as a mathematician with a PhD in logic (1974, ETHZ Zurich), in model theory, to be precise. I started publishing in Computer Science in 1980, in database theory, logic programming and finite model theory. For the last twenty years I mostly worked on applications of logic to combinatorics [7]. I have been teaching logic for Computer Science and other logic related courses for over thirty years at the Technion's Computer Science Department. I have designed the current version of the compulsory logic course, called *Sets and Logic for Computer Science*<sup>3</sup>. It consists of three hours per week frontal lectures and two hours per week tutorials in a thirteen week semester. It is given twice or even three times a year with close to 400 students enrolling each year. There are various teachers teaching the course, some of them leading researchers who use logic in their work, some of them colleagues who are well educated in logic and happen to like teaching logic, without logic playing a major part in their research. This makes introducing changes in the course more difficult. I tried and still try to shift the emphasis of this course. I lectured about the rationale behind this course and the changes I want to introduce at various conferences, formally at LPAR 2007, CiE 2008 [14, 16], and informally at Computer Science in Russia, Moscow 2008. I lectured about it also at ETH Zurich and the Technical University in Vienna. A journal version appeared as [15].

I have received many comments, criticisms, and encouragement<sup>4</sup>. In this paper I want to sharpen my point of view by addressing many of the critical comments I have received in the past.

## 2 Learning from the Past: Linear Algebra

We first look at the introduction of Linear Algebra into the curriculum of Mathematics and Physics students in the 1950ies. It took more than ten years to become a mainstream policy to teach Linear Algebra in the first year. We shall see in the sequel that logic courses in undergraduate Computer Science fail to achieve similar goals because they focus on the wrong issues.

The arguments for introducing Linear Algebra were twofold. Students should be exposed to Bourbaki-style abstract thinking and they should learn the tools needed later in courses like Numeric Analysis, Differential Equations, Functional Analysis, Mathematical Physics, and Quantum Physics. Linear Algebra seemed ideal for both purposes. On the abstract side, linear functions between vector spaces are distinguished from their representations via matrices over the underlying field. This is a first example where representation of (matrix-syntax) and being a linear function (function-semantics) is distinguished. Solving linear equations is a semantic problem, whereas computing a determinant is a syntactic method. The proposed course would have two goals: normal forms of matrix representations and the

<sup>3</sup> <http://webcourse.cs.technion.ac.il/234293/>

<sup>4</sup> I would like to thank Nadia Labai for her critical reading of this paper.

foundations of the determinant method. One really proves a *completeness theorem*: A system of linear equations has a unique solution (semantics) iff the determinant representing these equations does not vanish (syntax). The introduction of such a course was a big success for two reasons: It provided the student with a theoretical framework to be used over and over again. More importantly however, the concepts lived later on and were applicable in the most diverse situations. I have emphasized here the syntax/semantics dichotomy. The truth, however, is that most teachers of such a course are not aware of this distinction.

For some strange reasons, versions of Linear Algebra or Modern Algebra<sup>5</sup> are still taught almost the same way in the first year of a Computer Science curriculum. The reasons are strange, because it is not only conceivable but entirely normal that students of Computer Science will never encounter determinants in their entire three year program. If a student nevertheless encounters these concepts later on, be it in Numeric Analysis, Coding Theory or Complexity, the teacher feels obliged to develop the concepts from scratch. In other words such a course given in the first year is a total waste of time.

### 3 Theoretical Orientation vs Practical Knowledge

It is useful to distinguish between *theoretical orientation* and *practical knowledge*. Most electricians do have a lot of practical knowledge which they can apply when installing or repairing wiring and appliances. They may have a vague knowledge of the physics and electrodynamics on which their practical knowledge is based, but they do not have to understand the Maxwell equations. Their theoretical orientation is very limited. Structural engineers must have a very sophisticated practical knowledge of material science and applied mathematics, but again their theoretical orientation concerning the foundations of physics and real and complex analysis remains vague.

Our example of the Linear Algebra course is more balanced in this respect. It tries to convey the level of abstraction needed to understand (rather than use) the tools of matrix calculus, and it does also teach how to use those tools. In Linear Algebra courses for engineers the roles of understanding vs using may be shifted in favor of using.

So what does this mean for teaching Sets and Logic for Computer Science?

### 4 The Traditional Narrative

When logic and set theory courses were introduced in the Mathematics curriculum at about the same time as linear algebra the main purpose was theoretical orientation. The students should be exposed to the proposed solution of the so called crisis in the foundations of mathematics. The standard textbooks were [8] by P. Halmos for naive set theory and the books [18, 13, 6] by E. Mendelson, R.C. Lyndon and H. Enderton respectively, for mathematical logic. Halmos' book was reprinted in 1974, but now K. Devlin's [3] has taken its place. The books by Mendelson and Enderton are still reprinted and used. Lyndon's text teaches logic, provided the reader already knows abstract algebra. More modern textbooks I like are [5, 4]<sup>6</sup>

Naive set theory teaches the student to use set theory for defining ordinals and cardinals and their arithmetic. Sometimes Russels' paradox is discussed as making a problem, sometimes

<sup>5</sup> The original title of van der Waerden landmark monograph [2], where the word "modern" was dropped long ago.

<sup>6</sup> For mathematicians there are also the classics [20, 19, 17]. My true favorite is [1] by S. Adamowics and P. Zbierski.

as a hint that not all concepts can be formalized as sets. The axiom of choice and well-orderings play a prominent role. Very little of this, however, is relevant for our Computer Science undergraduates. I will briefly sketch what may be relevant for them in the next section.

The logic texts focus on First Order logic FOL. They define first order formulas and their interpretations. They give the semantic notions of logical consequence and logical equivalence. Then they give the syntactic notions of deduction rules and proofs (proof sequences). They may distinguish between natural deduction (Gentzen style) and Hilbert style deductions. They prove the completeness and compactness theorems, the Löwenheim-Skolem-Tarski theorems, and show that neither the real number field nor the arithmetic structure of the natural numbers can be characterized in First Order Logic. They may prove Gödel's incompleteness theorems. Again, this may give some theoretical orientation. But none of this is the *raison d'être* why our Computer Science undergraduates should learn logic. There is no practical knowledge gained from proving the Completeness and Compactness Theorems in full rigor.

The first proper modern logic monograph was [10] by D. Hilbert and W. Ackermann, which appeared first in 1928, and in English as [11] in 1950. One should emphasize that for Hilbert and Ackermann, logic was Second Order Logic, and what we call First Order Logic is called in their book the *restricted calculus*. They gave an axiomatization of the restricted calculus and asked whether their axiomatization is complete. K. Gödel, reading the book in its first year of publication, showed immediately that the answer was positive. As it follows from Gödel's incompleteness theorem that Second Order Logic has no recursive axiomatization, Second Order Logic was deemed to be too much like set theory, and First Order Logic took center stage. Still, the natural language to describe mathematics is second or even higher order logic<sup>7</sup>.

## 5 For Whom Should We Teach Sets and Logic in Computer Science?

We take our clue from the discussion about Linear Algebra. What we teach in Sets and Logic should be visibly used in the following courses:

- Data Structures and Algorithms
- Formal Languages, Automata Theory, and Computability
- Database Systems
- Graph Algorithms and Complexity
- Formal Methods and Verification, in all their ramifications
- Decision Procedures in Automated Theorem Proving

If the undergraduate curriculum contains in one form or another at least three of the above topics, a course like Sets and Logic should be taught. What remains to be discussed is the choice of topics and their emphasis.

## 6 The World of Sets

Our undergraduates do not need an introduction into cardinal arithmetic, but they do need an understanding and proficiency in handling statements like

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<sup>7</sup> I still was a witness in 1966 of an argument between P. Cohen and P. Bernays on whether Cohen had proved the independence of the Continuum Hypothesis from set theory. Bernays insisted that Hilbert's problem was not about first order provability in Zermelo-Fränkel set theory.

- An ordered pair  $\langle a, b \rangle$  is a set with the *basic property*  $\langle a, b \rangle = \langle a', b' \rangle$  iff  $a = a'$  and  $b = b'$ .
- There are various realizations of ordered pairs:
  - (i)  $\langle a, b \rangle_{Ku1} = \{\{a\}, \{a, b\}\}$ .
  - (ii)  $\langle a, b \rangle_{Ku2} = \{a, \{a, b\}\}$ .
  - (iii)  $\langle a, b \rangle_{Wie} = \{\{\{a\}, \emptyset\}, \{\{b\}\}\}$ .
- A finite automaton is a quintuple  $\langle S, \Sigma, \delta, s, F \rangle$  where  $S$  is a finite set (of states),  $\Sigma$  is a finite set of symbols (an alphabet),  $\delta : S \times \Sigma \rightarrow S$  is a (transition) function,  $s \in S$  is a starting state, and  $F \subseteq S$  is the subset of final (accepting) states.

Additionally, they do need a repertoire of set constructions<sup>8</sup> which allows them to construct sets which are realizations of the concepts we use in Computer Science:

- Sets, relations, functions, Cartesian products, infinite unions;
- Finite sets, countable sets, uncountable sets, equipotent sets;
- The set of words  $\Sigma^*$  over an alphabet  $\Sigma$  and the set of natural numbers  $\mathbb{N}$ .
- Inductively defined sets, such as the well-formed formulas in some formal language, etc.

Some of this material is usually covered in the beginning of high-level textbooks such as [12]. To the usual material, I would add without proof and discussion of the axiom of choice, also the statements:

- The Cartesian product of a non-empty (finite) family of non-empty sets is not empty.
- The union of countably many countable sets is countable.

However, I would advise not to talk about the axiom of choice or the well-ordering principle, and cardinalities of sets. For our purpose it suffices to talk about equipotent sets. Sets are finite, countable, or equipotent to some previously constructed set, such as the powerset of the natural numbers, are the set of function  $f : \mathbb{N} \rightarrow \{0, 1\}$ .

## 7 Side Effects

One aspect is never covered in the usual introduction to the world of sets: The realization of concepts as sets has **side effects**.

Side effects are properties of the realization which are not intended. In the example of the three realizations of ordered pairs,  $a$  is an element only of  $\langle a, b \rangle_{Ku2}$ ,  $b$  is not an element of any of them,  $\{a\}$  is an element of  $\langle a, b \rangle_{Ku1}$  only, and  $\{\{b\}\}$  is an element of  $\langle a, b \rangle_{Wie}$  only. The further difference between  $\langle a, b \rangle_{Ku1}$  and  $\langle a, b \rangle_{Ku2}$  is that to prove the basic property for the former, extensionality suffices, whereas for the latter the axiom of foundations is needed. This observation is needed to explain why  $\langle a, b \rangle_{Ku1}$  is preferable over  $\langle a, b \rangle_{Ku2}$ . It is also preferable over  $\langle a, b \rangle_{Wie}$ , because  $\langle a, b \rangle_{Wie}$  uses the empty set as part of its definition, whereas the other two versions use only curly brackets and the sets  $a$  and  $b$ .

We can look also at two definitions of natural numbers, both starting with the empty set realizing the element 0. For  $\mathbb{N}_{naive}$  we define the successor of  $n$  as  $\{n\}$ . For  $\mathbb{N}_{vNeu}$  we define the successor of  $n$  as  $n \cup \{n\}$ . They both satisfy the Peano Postulates. The side effects are that in the first every  $n \neq 0$  has exactly one element, whereas in the second every  $n$  is equipotent to the set of its predecessors.

The notion of side effects, as properties of a realization or implementation which are not intended, is central to understanding hardware and software systems. It can be easily explained in the world of sets, and should be a *leitmotiv* throughout all the courses taught.

<sup>8</sup> Basically these are the constructions needed to build the cumulative hierarchy

## 8 Syntax and Semantics

Having learned to master inductive definitions we should now introduce the syntax of logical formalisms. This looks like a further exercise in inductive definitions. We define the formalisms of Quantified Propositional Logic and prove the Unique Readability Theorem. The meaning of a formula is a Boolean function. We can introduce the notions of logical equivalence and consequence, and prove normal form theorems and quantifier elimination. We can introduce proof sequences and state, but not prove, the completeness theorem for the proof sequences introduced. Proving it requires too much time which we need for other purposes.

As said before, Second Order Logic SOL is the logic which allows us to talk about most important concepts, say in graph theory or formal language theory. Therefore we introduce the syntax of SOL right away. We start with a vocabulary (set of relation and function symbols including constant functions which are constants). The meaning (interpretation) of a vocabulary  $\tau$  is a  $\tau$ -structure. The meaning of an SOL( $\tau$ )-formula with free first order variables is a relation in a  $\tau$ -structure. If there are no free variables, it is a Boolean value, but it is preferable to treat this as a special case and keep the case with free variables in the center of our attention. If there are free second order variables then the meaning is given by sets of relations.

## 9 Read and Write

Before we embark on proving meta-theorems the students should have some minimal proficiency in reading and writing SOL-formulas. The mathematics students learning logic can be assumed to be familiar with mathematical statements from solving equations and from analytic geometry. For them, the notion of the geometric lieu of all points satisfying a set of equations might come natural. The Computer Science students have no such background. I am always perplexed to see how difficult it is, even for advanced Computer Science students, to correctly write down an SOL-formula expressing that a graph is connected, or Hamiltonian. The difficulty is real (and mathematical) when trying to formulate that a graph is planar, because we have to use Kuratowski's or Wagner's Theorem. Once the students have written down a formula which supposedly expresses what they had in mind, it remains difficult for the students to read their own formula in order to check its correctness.

Practicing reading and writing skills by drawing graphs and finding FOL formulas which distinguish the graphs, or internalizing that isomorphic structures cannot be distinguished, is a prerequisite for all further developments. This brings us back to side effects. It is also a side effect that two isomorphic structures are different from the point of view of their definition in the language of sets.

## 10 Pebble Games Help US Understand Quantification

Students encounter great difficulties in grasping the order of quantification even in FOL. It might be useful to introduce pebble games at an early stage. They are very visual and leave an impression on the students. Reading off the winning strategy from formulas in prenex normal form is very visual and easy to understand and to prove. So if a formula with  $n$  quantifiers distinguishes two structures, player I has a winning strategy. The converse, if no formula with  $n$  quantifiers distinguishes two structures, then player II has a winning strategy, can be stated without proof. The easy part can be used to show that

- One needs at least  $n$  existential quantifiers to say that there are at least  $n$  elements in the structure over the empty vocabulary.
- One cannot express that there is an even number of elements in the structure over the empty vocabulary.

If we look at a the graph of the gender-free parent relation, there are natural concepts like grandparents, siblings, cousins, and even third-cousins twice removed, which can be expressed on FOL. Other concepts, like being an ancestor, or the requirement that the parent relation be cycle-free, are not expressible in FOL, even if restricted to finite structures. Again, to show the non-definability in FOL one can use the pebble games.

One should also prove that there are  $\forall\exists$ -formulas in FOL which are not equivalent to any  $\exists\forall$ -formulas, and vice versa. Again this is not difficult using the properties that universal formulas are preserved under substructures (which can be modified to  $\exists\forall$ -formulas), and that  $\forall\exists$ -formulas are preserved under unions of chains.

All this is more relevant for the later courses than proving the completeness or compactness theorem.

## 11 Definability and Non-definability

Definability is a central concept of Logic. We can define concepts in the language of sets using the membership relation only. These are the set-definable concepts. The realization of these concepts as sets have many properties expressible in the language of sets from which we want to abstract. By defining concepts as  $\tau$ -structures we fix the level of abstraction, and distinguish between the essentials and the side effects.

Every SOL-definable concept is also set-definable, but the converse is not true. Topological spaces are good examples for mathematicians. They are not  $\tau$ -structures in the sense we have in mind. For our Computer Science undergraduates one can explain the difference between set-definability and SOL-definability by anticipating the notion of computable languages (sets of words): Every computable language is set-definable, but SOL-definable languages have bounded complexity. FOL-definable concepts are computationally even simpler. Logical formalisms are used to compute definable concepts. We choose the formalisms to be expressive enough for use in a particular application, but to be computationally feasible enough to allow implementation.

## 12 Conclusions

We have suggested that the meta-mathematical narrative of the traditional logic courses for Computer Science is missing its purpose and contributes to the disappearance of logic from the undergraduate curriculum.

We have emphasized that the world of sets is used for modeling concepts of Computer Science using simple set constructions and inductive definitions. It is also used to prove properties of these concepts. We suggested to introduce, first as an example of long inductive definitions, quantified propositional logic and its meaning, given by Boolean functions. This is used to develop the basic concepts of logic, logical equivalence and consequence. We suggested as a next step to introduce Second Order Logic, and leave First Order Logic as a special case. We put the emphasis on teaching how to read and write properties of the modeled concepts, and to concentrate on the expressive power of Second Order Logic and its fragments. Finally we suggested to concentrate on tools to prove non-definability such as pebble games and preservation properties.

So where are all the beautiful theorems to be taught? In follow up courses when needed. Completeness and incompleteness in a course on decision procedures of logical systems, or in a course on proof theory, compactness in a course on model theory for mathematicians, or in a course on the foundations of logic programming. None of these will be undergraduate courses. In contrast to this, what we suggest to teach is used in

- Data Structures and Algorithms
- Formal Languages, Automata Theory and Computability
- Database Systems
- Graph Algorithms and Complexity
- Formal Methods and Verification, in all their ramifications
- Decision Procedures in Automated Theorem Proving

Explaining Completeness and Compactness can be viewed as part of the *theoretical orientation*. Proving these theorems on the expense of using the meaning function to teach reading and writing SOL and FOL formulas amounts to depriving the students of the practical knowledge they need later on.

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