Lecture 6
What did we do so far ?

- We have learned how to build and use sets to model objects used in computation.
- We have learned how to use inductive definitions and induction proofs.
- We have built the set **HF**. Most of our future objects will be either elements or relations on **HF**.
- We have learned that there are many sets of finite size.
- We have seen that there are many versions of infinite sets, countable and uncountable.

Outline of lecture 6

- We shall informally introduce Propositional Logic.
- We define the **syntax** of Propositional Logic as the set of words **WFF** which model the formulas of Propositional Logic.
- We formulate the property of unique readability of formulas.
- We define the semantics or meaning of formulas of **WFF** as boolean functions using the rank of formulas.
- We introduce the basic semantic concepts **satisfiability, logical equivalence** and **logical consequence** in Lecture 7

Propositional connectives

We want to analyse the (universal) words

and, or, not
and
if ... then
and
if and only if or iff.

They are called propositional connectives.

They connect formulas of **WFF**,
also called sentences or propositions.

Atomic (simple) propositions

P_1: There will be elections.

P_2: Dinosaurs drink beer.

P_3: $a \in X$.

P_4: $F : A \to B$ is a one-one function.

P_5: There are no sets A such that $\mathbb{N} \prec A \prec \wp(\mathbb{N})$.

Atomic propositions are **true** or **false**
but we may not know which of the two.

Propositions (statements)

From atomic propositions we may form

If
((if P_1 **then** P_2 **) and (** P_1 **or** P_2 **))**
then P_2.

If
(
(if P_3 **then** P_4 **) and (if** P_4 **then not** P_3 **)**
)
then P_5.

Read these aloud with the full text of $P_1, \ldots, P_5$.

Can you analyse their meaning?

Why do we model Propositional Logic in the world of sets?

- To clarify the meaning of complex sentences.
- To make the meaning of complex sentences computable.
- To computerize the manipulation of formulas of propositional logic.
- To enable computer aided and automated theorem proving (in mathematics and other exact sciences).

Back to outline of Lecture 6

The syntax of propositional logic (An inductive definition)

Let

$$Symb = \{\wedge, \vee, \rightarrow, \neg, \mathbf{T}, \mathbf{F}, (,)\}$$

and

$$Var = \{p_0, p_1, p_2, ..., p_i, ...\}$$

where $i \in \mathbb{N}$.

We treat the elements of Var also like atoms. This means they are coded as sets, but we shall not use any properties of this coding.

$Symb$ is called the set of logical symbols and parentheses.

$\wedge$ is pronounced *and*, $\vee$ is pronounced *or*,
$\neg$ is pronounced *not*, $\rightarrow$ is pronounced *arrow*,
$\mathbf{T}$ is pronounced *true*, $\mathbf{F}$ is pronounced *false*.

Var is called the set of propositional variables.

Definition 1 (3.1.1)

The set of well formed formulas **WFF** *as words*

The set of *well formed formulas* **WFF** is a subset of $(Symb \cup Var)^*$, the finite words over $Symb \cup Var$, defined inductively as follows:

Basis: (Atomic formulas of **WFF**).
(i) For each $i \in \mathbb{N}$ p_i is in **WFF**.
(ii) $\mathbf{F} \in$ **WFF**.
(iii) $\mathbf{T} \in$ **WFF**.

Closure:
(i) If ϕ_1, ϕ_2 are in **WFF**, so is $(\phi_1 \wedge \phi_2)$;
(ii) If ϕ_1, ϕ_2 are in **WFF**, so is $(\phi_1 \vee \phi_2)$;
(iii) If ϕ_1, ϕ_2 are in **WFF**, so is $(\phi_1 \rightarrow \phi_2)$;
(iv) If ϕ is in **WFF**, so is $\neg\phi$.

Some Greek letters

We shall always use lower case Greek letters to denote elements of **WFF**.

ϕ is pronounced (in Hebrew) *Fee*.
ϕ_0 is pronounced *Fee zero*.
ϕ_n is pronounced *Fee enn*.

ψ is pronounced *Psee*.
ψ_0 is pronounced *Psee zero*.
ψ_n is pronounced *psee enn*.

χ is pronounced *Khee*.
θ is pronounced *Theta*.

ϕ and $\emptyset$ are different symbols.

We shall sometimes write

$$\phi(p_1, p_2, ..., p_n)$$

meaning that all the variables occurring in the word ϕ are among $p_1, p_2, ..., p_n$.

Examples 2 (3.1.2)

Well formed formulas

ϕ_1, ϕ_2 and ϕ_3 are formulas of **WFF**:

$$\phi_1 : ((p_1 \to \mathbf{T}) \wedge \neg(\neg p_3 \vee p_1))$$

$$\phi_2 : (((p_1 \to p_2) \to p_1) \to p_3)$$

$$\phi_3 : \neg(\neg p_1 \wedge p_2), \quad \phi_4 : (\neg\neg p_1 \vee p_2)$$

We also write (as abreviations):

$$((\phi_1 \wedge \phi_1) \to \phi_3)$$

Here ϕ_1, ϕ_2, ϕ_3 are subroutine calls in the inductive definition.

Examples 3

Words which are not well formed formulas

Not well formed formulas of **WFF**:

$$(((\neg)), \quad (\neg p_1)), \quad (\neg p_1),$$

$$p_1 \wedge p_2, \quad (p_1 \wedge p_2, \quad (p_1 \wedge p_2)), \quad p_1 p_2 \neg)).$$

$$(p_1 \to \mathbf{T}) \wedge \neg(\neg p_3 \vee p_1)$$

$$((p_1 \to p_2) \to p_1 p_3) \to p_3)$$

Definition 4

Formulas of $\mathbf{WFF}_{\rightarrow,\mathbf{False}}$

The set $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}} \subseteq \mathbf{WFF}$ is defined similarly but using only (i) and (ii) of the basis and (iii) of the closure condition.

Well formed formulas of $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$:

$$(((p_1 \rightarrow \mathbf{F}) \rightarrow p_1) \rightarrow \mathbf{F}),$$

$$((p_1 \rightarrow \mathbf{F}) \rightarrow (p_1 \rightarrow \mathbf{F})).$$

Not well formed formulas of $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$:

$$p_1 \wedge p_2, (p_1 \wedge p_2), (\mathbf{T} \vee \mathbf{F})$$

Reading and writing formulas

Given a word $w \in (Symb \cup Var)^*$
we show that $w \in$ **WFF** by exhibiting a

generating sequence.

$p_1, \mathbf{T}, p_3$
$\neg p_3$, $(p_1 \to \mathbf{T})$, $(\neg p_3 \vee p_1)$,
$\neg(\neg p_3 \vee p_1)$

hence

$$((p_1 \to \mathbf{T}) \wedge \neg(\neg p_3 \vee p_1)) \in \textbf{WFF}$$

We want to define various relations and functions inductively on well-formed formulas **WFF**.

Definition 5 (3.1.3)

Well formed tree formulas **WFTF**

Basis: $Var \subseteq$ **WFTF** and $\mathbf{T}, \mathbf{F} \in$ **WFTF**.
They are trees consisting of a single node.

Closure: (i) If $T_1, T_2 \in$ **WFTF** then the tree with root $\wedge$ and T_1 as its left son and T_2 as its right son is also in **WFTF**.

(ii) If $T_1, T_2 \in$ **WFTF** then the tree with root $\vee$ and T_1 as its left son and T_2 as its right son is also in **WFTF**.

(iii) If $T_1, T_2 \in$ **WFTF** then the tree with root $\rightarrow$ and T_1 as its left son and T_2 as its right son is also in **WFTF**.

(iv) If $T_1 \in$ **WFTF** then the tree with root $\neg$ and the only subtree T_1 as its son is also in **WFTF**.

Definition 6

Alternative view of **WFTF**

We treat $\neg$ as a unary function and
$\wedge, \vee$ and $\rightarrow$ as binary functions on **WFTF**.

Basis: $Var \subseteq$ **WFTF** and $\mathbf{T}, \mathbf{F} \in$ **WFTF**.

Closure: (i) If $T_1, T_2 \in$ **WFTF** then $\wedge(T_1, T_2) \in$ **WFTF**.

(ii) If $T_1, T_2 \in$ **WFTF** then $\vee(T_1, T_2) \in$ **WFTF**.

(iii) If $T_1, T_2 \in$ **WFTF** then $\rightarrow (T_1, T_2) \in$ **WFTF**.

(iv) If $T_1 \in$ **WFTF** then $\neg(T_1) \in$ **WFTF**.

WFTF is the smallest set containing
$Var \cup \{\mathbf{T}, \mathbf{F}\}$ which is closed under the functions
$\neg, \wedge, \vee$ and $\rightarrow$.

So we can use inductive definitions and proofs on **WFTF**.

Back to outline of Lecture 6

Writing formulas

Let $T \in$ **WFTF**.

We can think of writing a well formed formula ϕ as the process of obtaining a word $write(T)$ from a tree T.

> We start at the leaves and just write them down as words.
>
> If we are at a node labeled by a symbol $\Box$ from $\{\wedge, \vee, \rightarrow\}$ and two subtrees T_1, T_2 we add the parentheses and write the word
>
> $$(write(T_1)\Box write(T_2))$$
>
> If a node is labeled $\neg$ followed by a subtree T we write the word
>
> $$\neg write(T)$$

We easily observe that $write(T) \in$ **WFF**,
and that writing is unique:
There is exactly one word ϕ which can be obtained from T in the above way.

Reading formulas

We can think of reading a formula ϕ as the process of obtaining a tree T from the word ϕ.

This is a more complicated process and has to be proven by induction.

Theorem 7 (3.1.4) Unique readability of WFF

*For every well formed formula $\phi \in$ **WFF** there is exactly one labelled tree T such that $write(T) = \phi$.*

The proof is given in one of the Tirgulim.
Skip the outline of the proof

Proof of unique readability, I

Tirgul

The proof uses the Lemmas 8-14 below.

The details of the proof of the theorem and the lemmas are left as an exercise.

The first four lemmas can be proved independently.

To prove lemma 13 one needs all the previous lemmas.

For lemma 14 one needs only lemma 13.

Finally, to prove the theorem, one needs only lemma 14.

Proof of unique readability, II

Tirgul

Lemma 8

If $\phi \in$ **WFF** *then* $\phi \neq \epsilon$*, i.e. is not the empty word, and contains at least one variable* $p_i \in Var$*, or one of the constants* **T,F***.*

Lemma 9

Let $\phi \in$ **WFF***.*
Then the number of left parenthesis in ϕ *equals the number of right parentheses in* ϕ*.*

Proof of unique readability, III

Tirgul

Definition 10

A word $\alpha \neq \epsilon$ is a **proper initial segment** *of a word ϕ if there exists a word β such that $\beta \neq \epsilon$ and $\phi = \alpha \circ \beta$.*

Lemma 11

*Let $\phi \in$ **WFF** and α be a proper initial segment of the word ϕ. Then either α contains no parentheses or the number of left parentheses of α is bigger than the number of its right parentheses.*

Proof of unique readability, IV

Tirgul

Lemma 12

*Let $\phi \in$ **WFF** and α be an initial segment of the word ϕ.*
If $\alpha \neq \phi$ and has no parentheses then $\alpha \in \{\neg\}^*$.

Lemma 13

*Let $\phi \in$ **WFF** and α be a proper initial segment of the word ϕ.*
*Then $\alpha \notin$ **WFF**.*

Proof of unique readability, V

Tirgul

Lemma 14

Let $\phi, \alpha, \alpha_1, \beta, \beta_1 \in$ **WFF**.

(i) *Let* $\bullet, \bullet_1 \in \{\wedge, \vee, \rightarrow\}$.
If $\phi = (\alpha \bullet \beta$ *and* $\phi = (\alpha_1 \bullet_1 \beta_1$ *then* $\alpha = \alpha_1, \beta = \beta_1$ *and* $\bullet = \bullet_1$.

(ii) *If* $\phi = \neg\alpha$ *and* $\phi = \neg\beta$ *then* $\alpha = \beta$.
Furthermore, there are no α_1, β_1 *and no* $\bullet \in \{\wedge, \vee, \rightarrow\}$ *such that* $\phi = (\alpha_1 \bullet \beta_1$.

Combining the preceeding lemmas gives a proof of the above theorem. Q.E.D.

Homework

- Change the definition of **WFF** such that no parentheses are used and show that the Unique Readability Theorem fails for the resulting definition.

- Polish notation of formulas. We define POL inductively:

  Basis: $Var \cup \{\mathbf{T}, \mathbf{F}\} \in POL$.

  Closure: If $u, v \in POL$ so are
  $\neg v, \wedge uv, \vee uv$ and $\rightarrow uv \in POL$.

  Show that the Unique Readability Theorem is true for this definition.

Back to the Unique Readability Theorem

Back to outline of Lecture 6

Rank of a Formula

The inductive definition of **WFF** suggests that formulas are defined in stages $\mathbf{WFF}_n$.

Inductive definitions are bottom up constructions.

The Unique Readibility Theorem allows us to view formulas also top down without ambiguity.

We want to make this precise.

Given ϕ, what is the first $n \in \mathbb{N}$ such that $\phi \in \mathbf{WFF}_n$?

Definition 15 (3.1.13)

Rank of a formula

Let $rank$ be a function $rank : \mathbf{WFF} \to \mathbb{N}$.

Basis: $\mathbf{WFF}_0 = \{\mathbf{T}, \mathbf{F}\} \cup Var$.
If $\phi \in \mathbf{WFF}_0$ then $rank(\phi) = 0$.
We call the formulas of $\mathbf{WFF}_0$ also
atomic formulas.

Closure: Let $\phi_1, \phi_2 \in \mathbf{WFF}_n$. Then
ϕ_1, $(\phi_1 \wedge \phi_2)$, $(\phi_1 \vee \phi_2)$, $(\phi_1 \to \phi_2)$ and $\neg\phi_1$ are in $\mathbf{WFF}_{n+1}$.

$rank(\phi_1)$ is defined to be the smallest $n \in \mathbb{N}$
such that $\phi_1 \in \mathbf{WFF}_n$.

Proposition-Exercise 16 (3.1.14)

Properties of the rank

Compute the rank of

$$(((p_1 \to p_2) \to p_1) \to p_3)$$

$$(\neg(\neg p_1 \wedge p_2) \to (\neg\neg p_1 \vee p_2))$$

Show the following:

(i) $\mathbf{WFF}_n \subseteq \mathbf{WFF}_{n+1}$;

(ii) $rank(\phi_1 \wedge \phi_2) = 1 + max\{rank(\phi_1), rank(\phi_2)\}$;

(iii) $rank(\phi_1 \vee \phi_2) = 1 + max\{rank(\phi_1), rank(\phi_2)\}$;

(iv) $rank(\phi_1 \to \phi_2) = 1 + max\{rank(\phi_1), rank(\phi_2)\}$;

(v) $rank(\neg\phi_1) = 1 + rank(\phi)$.

Truth table semantics of propositional logic

- The **meaning** of a well formed formula of **WFF** is a an element of the set $\{0, 1\}$, where we think of 0 as "false" and 1 as "true".

- Note that the set $\{0, 1\}$ is a subset of $\mathbb{N}$ and the intended interpretation of its elements as "true" and "false" is outside the scope of our mathematical framework.

- We can think of the truth tables as

  behavioural descriptions of boolean circuits.

  The truth tables of a $\vee$, $\wedge, \neg$ describe the behaviour of an **or**–gate, an **and**–gate, **not**–gate respectively.

The meaning of a proposition

The definition of a **meaning function** given in in the language of sets is the goal of this subsection.

We shall do this in three stages:

- *k-ary truth tables,*
  which are functions $TT : \{0,1\}^k \to \{0,1\}$

- *The set Ass of assignments,*
  which are functions $z : Var \to \{0,1\}$

- *The meaning function,*
  which is a function

$$M : \mathbf{WFF} \times Ass \to \{0,1\}$$

Definition 17 (3.1.19)

Truth tables

Let $n \in \mathbb{N}$. An **n-ary truth table** TT is a function $TT : \{0,1\}^n \to \{0,1\}$.

With each symbol $\wedge, \vee, \to$ we shall associate binary truth tables $TT_\wedge, TT_\vee, TT_\to$ respectively in the following way:

$TT_\wedge$ is given by
$TT_\wedge(0,0) = 0$, $TT_\wedge(0,1) = 0$,
$TT_\wedge(1,0) = 0$, $TT_\wedge(1,1) = 1$,

$TT_\vee$ is given by
$TT_\vee(0,0) = 0$, $TT_\vee(0,1) = 1$,
$TT_\vee(1,0) = 1$, $TT_\vee(1,1) = 1$,

Definition 18 (3.1.19, contd)

Truth tables (contd)

$TT_{\rightarrow}$ is given by
$TT_{\rightarrow}(0,0) = 1$, $TT_{\rightarrow}(0,1) = 1$,
$TT_{\rightarrow}(1,0) = 0$, $TT_{\rightarrow}(1,1) = 1$,

$TT_{\neg}$ is given by
$TT_{\neg}(0) = 1$ and $TT_{\neg}(1) = 0$.

$TT_{\mathbf{T}}$ is given by
$TT_{\mathbf{T}} = 1$ and

$TT_{\mathbf{F}}$ is given by
$TT_{\mathbf{F}} = 0$.

Definition 19 (3.1.21)

Truth assignments

A *(propositional) truth assignment* is a function

$$z : Var \to \{0, 1\}.$$

We denote by Ass the set $\{0, 1\}^{Var}$ of all truth assignments.

We can think of the variables in Var as registers,
and of z as a function reading the contents
of the registers in a current state.

Definition 20 (3.1.23)

The meaning function M

A **meaning function** M is a function $M : \mathbf{WFF} \times Ass \to \{0, 1\}$.

We shall denote by M_{PL} the meaning function for Propositional Logic defined inductively as follows:

Basis:

$M_{PL}(p_i, z) = z(p_i)$;

$M_{PL}(\mathbf{T}, z) = TT_{\mathbf{T}} = 1$;

$M_{PL}(\mathbf{F}, z) = TT_{\mathbf{F}} = 0$.

Definition 21 (3.1.23, contd)

The meaning function M (contd)

Closure:

$$M_{PL}((\phi_1 \wedge \phi_2), z) = TT_{\wedge}(M_{PL}(\phi_1, z), M_{PL}(\phi_2, z));$$

$$M_{PL}((\phi_1 \vee \phi_2), z) = TT_{\vee}(M_{PL}(\phi_1, z), M_{PL}(\phi_2, z));$$

$$M_{PL}((\phi_1 \rightarrow \phi_2), z) = TT_{\rightarrow}(M_{PL}(\phi_1, z), M_{PL}(\phi_2, z));$$

$$M_{PL}(\neg\phi, z) = TT_{\neg}(M_{PL}(\phi, z)).$$

Example

We let z be given by

$$z(p_0) = 1, z(p_1) = 0, z(p_2) = 1, z(p_3) = 0, z(p_4) = 1, \ldots$$

and compute

$$M\left(((p_1 \to \mathbf{T}) \wedge \neg(\neg p_3 \vee p_1)), z\right)$$

$M(p_1, z) = 0,\ M(p_3, z) = 0,$
$M(\neg p_3, z) = 1,$
$M((p_1 \to \mathbf{T}), z) = 1,$
$M((\neg p_3 \vee p_1), z) = 1,\ M(\neg(\neg p_3 \vee p_1), z) = 0,$
$M\left(((p_1 \to \mathbf{T}) \wedge \neg(\neg p_3 \vee p_1)), z\right) = 0$

Similarly, compute:

$$M\left((((p_1 \to p_2) \to p_1) \to p_3), z\right)$$

$M(p_1, z) = 0,\ M(p_2, z) = 1,\ M(p_3, z) = 0,$
$M((p_1 \to p_2), z) = 1,$
$M(((p_1 \to p_2) \to p_1), z) = 0,$
$M((((p_1 \to p_2) \to p_1) \to p_3), z) = 1$

Remarks

- M_{PL} defines also a meaning function $M_{\{\rightarrow,\mathbf{F}\}}$ for the formulas of $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

  As $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}} \subseteq \mathbf{WFF}$ we set

  $$M_{\{\rightarrow,\mathbf{F}\}} = M_{PL}|\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}} \times Ass$$

  the restriction of M_{PL} to $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}} \times Ass$.

- The definition of M_{PL} rests on the Unique Readibility Theorem for the formulas of **WFF**.

  It relies not on the word ϕ but on the unique T_ϕ such that $write(T_\phi) = \phi$.

Examples 22 (3.1.25)

Exercises

(i) Choose several $\phi \in$ **WFF** and propositional assignments z and compute $M_{PL}(\phi, z)$.

(ii) Let $\phi, \psi \in$ **WFF** and $z \in Ass$. Show:

(ii.a) If $\phi = (\phi_1 \to \phi_1)$ the $M_{PL}(\phi, z) = 1$.

(ii.b) if $\phi = (\phi_1 \wedge \neg\phi_1)$ then $M_{PL}(\phi, z) = 0$.

(ii.c) If $\phi = (\neg\phi_1 \vee \phi_2)$ and $\psi = (\phi_1 \to \phi_2)$ then $M_{PL}(\phi, z) = M_{PL}(\psi, z)$.

Homework

Finite dependency of the meaning function

It is obvious from the definitions, that the function M_{PL} only depends on finitely many values of z. The next proposition makes this precise.

Prove the following (by induction);

Proposition 23

*Let $\phi \in$ **WFF** be a formula with all its propositional variables in the set $\{p_1, p_2, ..., p_n\}$.*
Let z_1 and z_2 be two propositional assignments such that for every $i \leq n$ $z_1(p_i) = z_2(p_i)$.

Then $M_{PL}(\phi, z_1) = M_{PL}(\phi, z_2)$.

Proof of Proposition 23:

We prove by induction:

Basis:
If $\phi \in \mathbf{WFF}_0$ then $\phi = p_i$ for some $i \leq n$
or $\phi = \mathbf{T}$ or $\phi = \mathbf{F}$.
In all these cases $M_{PL}(\phi, z)$ depends only on the value of $z(p_i)$ or is constant.

Closure:
If $\phi \in \mathbf{WFF}_{n+1}$ and the proposition is true for all $\phi_1, \phi_2 \in \mathbf{WFF}_i$ $i \leq n$,
we have four cases.

Let $\phi = (\phi_1 \wedge \phi_2)$.
As $M_{PL}((\phi_1 \wedge \phi_2), z) = TT_\wedge(M_{PL}(\phi_1, z), M_{PL}(\phi_2, z))$
and $TT_\wedge$ does not depend on z,
the proposition is also true for ϕ.

The cases for $\vee, \rightarrow$, and $\neg$ are left to the reader.

Definition 24 (3.1.27)

Truth table associated with ϕ

We associate with each well formed formula $\phi \in$ **WFF** a truth table TT_ϕ in the following way:

Let $\phi \in$ **WFF** and let $p_{i_1}, p_{i_2}, ..., p_{i_n}$ be all the variables occuring in ϕ.

Let $TT_\phi : \{0,1\}^n \to \{0,1\}$ the truth table defined by

$$TT_\phi(x_1, x_2, ..., x_n) = M_{PL}(\phi, z)$$

with $z(p_{i_j}) = x_j$ for $j = 1, 2, \ldots, n$.

The finite dependency of the meaning function assures us that this is indeed a function in n variables.

Examples of truth tables, I

Let

$$T_{Scheffer}(x_1, x_2) = 1 \text{ iff } x_1 = x_2 = 0$$

and

$$\phi_{Scheffer} = (\neg\, p_1 \wedge \neg\, p_2).$$

Then $T_{Scheffer} = T_{\phi_{Scheffer}}$.

Examples of truth tables, II

Let

$$T_1(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{else} \end{cases}$$

and

$$\phi_1(p_1, p_2) = ((p_1 \wedge p_2) \vee (\neg\, p_1 \wedge \neg\, p_2))$$

Then $T_1 = T_{\phi_1}$.

Examples of truth tables, III

Let

$$T_2(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 = x_2 \text{ and } x_3 = 0 \\ 1 & \text{if } x_3 = 1 \\ 0 & \text{else} \end{cases}$$

and

$$\phi_2(p_1, p_2, p_3) = ((\neg\, p_3 \wedge ((p_1 \wedge p_2) \vee (\neg\, p_1 \wedge \neg\, p_2))) \vee p_3)$$

Then $T_2 = T_{\phi_2}$.

Is every truth table TT obtainable from a formula ϕ as TT_ϕ ?

Let TT be an n-ary truth table.

Does there exists a formula $\phi \in$ **WFF** such that $TT = TT_\phi$?

A positive answer to this question, would give us a justification for the choice of the basic truth tables

$$TT_\wedge, TT_\vee, TT_\neg$$

underlying the semantics of **WFF**.

Theorem 25 (3.1.28)

Functional completeness

Let TT be an n-ary truth table. Then there exists a formula $\phi \in \mathbf{WFF}$ such that $TT = TT_\phi$.

The formula can actually be found in $\mathbf{WFF}_{\rightarrow,\mathbf{F}}$ or in $\mathbf{WFF}_{\vee,\neg}$

Proof of Theorem 25

For $n = 0$ there are two constant truth tables.

For $n > 0$, let

$$x = \langle x_1, x_2, ..., x_n \rangle \in \{0, 1\}^n$$

such that $TT(x) = 1$.

Let

$$\ell_i = \begin{cases} p_i & \text{if } x_i = 1 \\ \neg p_i & \text{if } x_i = 0. \end{cases}$$

Let C_x be the conjunction of all the ℓ_i,

$C_x = ((...(l_1 \wedge l_2) \wedge \ldots \wedge l_n)$.

Now let ϕ be the disjunction of all the C_x such that $TT(x) = 1$.

It is now easy to verify that $TT_\phi = TT$. Q.E.D.

Tirgul: $\mathbf{WFF}_S$

We now generalize our notion of well formed formulas.
This will **not** be used in the sequel, but it is useful for checkin whether one understands the material presented so far.

Definition 26 (The set of well formed formulas $\mathbf{WFF}_S$:)
Let $S = \{s_1, s_2, ..., s_m\}$ be a set of symbols and $n(s_i) \in \mathbb{N}$ be a natural number called the arity *of s_i.*
Let $S_1 = S \cup \{(, ;,)\}$.
Let Var be the set of propositional variables.

$\mathbf{WFF}_S$ *is inductively defined as a subset of $(S_1 \cup Var)^*$.*

Basis:
$Var \subseteq \mathbf{WFF}_S$.

Closure:
If $s_i \in S$ and $n(s_i) = k$ and $\phi_1, \phi_2, ..., \phi_k \in \mathbf{WFF}_S$ then $s_i(\phi_1; \phi_2; ...; \phi_k) \in \mathbf{WFF}_S$.

Tirgul: $\mathbf{WFF}_S$ (continued)

Example 27 (Exercise:)
Let $S = \{\mathbf{F}, \neg, \wedge, kuku\}$
with $n(\mathbf{F}) = 0, n(\neg) = 1, n(\wedge) = 2$,
and $n(kuku) = 3$.

Write some formulas of $\mathbf{WFF}_S$.

Theorem 28 (Exercise:)
Formulate and prove the Unique Readibility Theorem for $\mathbf{WFF}_S$.

The following will be useful later:

Theorem 29
Let S *be countable. Then* $\mathbf{WFF}_S$ *is countable.*

Tirgul: **WFF**$_S$ (continued)

Definition 30 (Semantics for WFF$_S$:)
Let $S = \{s_1, s_2, ..., s_n\}$ be a set of symbols with arities $n(s_i) \in \mathbb{N}$.

For each $i \leq n$ let TT_i be an $n(i)$-ary truth table.

Let z be a propositional assignement.
We define a meaning function M_S inductively as follows:

Basis:
$M_S(\mathbf{T}, z) = 1$,
$M_S(\mathbf{F}, z) = 0$ *and*
$M_S(p_i, z) = z(p_i)$.

Closure:
*For every $i \leq n$ with $n(s_i) = k$ and $\phi_1, \phi_2, ..., \phi_k \in$ **WFF**$_S$*

$$M_S(s_i(\phi_1; \phi_2; ...; \phi_k), z) = TT_i(M_S(\phi_1), M_S(\phi_2), ..., M_S(\phi_k))$$

Tirgul: $\mathbf{WFF}_S$ (continued)

Let $S = \{\mathbf{F}, \neg, \wedge, kuku\}$
with $n(\mathbf{F}) = 0, n(\neg) = 1, n(\wedge) = 2$,
and $n(kuku) = 3$.

Let TT_{kuku} be an 3-ary truth table.

Show the following by induction.

Theorem 31
For every formula $\phi \in \mathbf{WFF}_S$
there is a formula $\psi \in \mathbf{WFF}$ such that for every assignment z we have

$$M_S(\phi, z) = M(\psi, z)$$