Lecture 8-9
What did we do so far ?

- Satisfiability, validity
- Logical equivalence
- Normal forms (**NNF**, **CNF**, **DNF**)
- Checking logical equivalence

To outline of Lecture 8-9

Outline of lecture 8 and 9

- Deduction systems
- Proof sequences
- Soundness and completeness of proof systems
- Deduction theorem
- Completeness of proof system
- Compactness
- Definability

Deduction Systems

We present a method of deduction:

Proof sequences

(Hilbert style deduction)

It is supposed to model human reasoning.

This is an **exageration**:
In the best case it models the way mathematicians and other educated people **should** write down their arguments, when pressed to do so.

In short, it models a stylized form of human reasoning.

In this sense it is quite user friendly.

Back to outline of Lecture 8-9

Deduction Systems: Soundness and completeness

Let $\mathbf{WFF}_{myown} \subseteq \mathbf{WFF}$ be such that for every $\phi \in WFF$ there is a $\psi \in WFF_{myown}$ with ϕ logically equivalent to ψ.

Recall: For $\Sigma \subseteq \mathbf{WFF}_{myown}$ we set

$$Con(\Sigma) = \{\phi \in \mathbf{WFF}_{myown} : \Sigma \models \phi\}$$

An inductive definition of $Ded(\Sigma)$ for $Con(\Sigma)$ is **sound** (for $Con(\Sigma)$) if

$$Ded(\Sigma) \subseteq Con(\Sigma).$$

It is **complete** (for $Con(\Sigma)$) if

$$Con(\Sigma) \subseteq Ded(\Sigma).$$

Many Deduction Systems

There are many

sound and complete

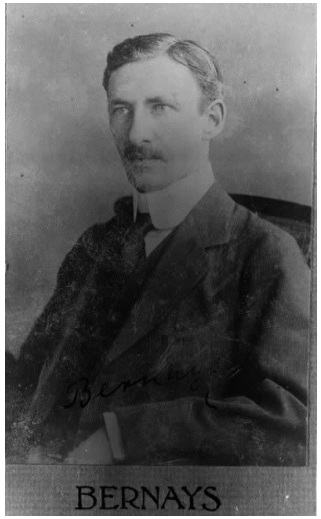
inductive definitions of $Con(\Sigma)$.

Natural deduction, Gentzen style:
Trivial basis ($\Sigma \subseteq Ded(\Sigma)$)
Many cases in the closure.
Close to mathematical reasoning

Hilbert style:
Many cases in the basis,
only one case (Modus ponens) in the closure
Not so close to mathematical reasoning

minimal Hilbert style, our systems:
Few operators, small basis
only one case (Modus ponens) in the closure
Good for analysing the system.

David Hilbert and Gerhard Gentzen



David Hilbert 1862-1943

Gerhard Gentzen 1909-1945

Paul Bernays 1888-1977

P. Bernays was the real logician
working for D. Hilbert and supervising G. Gentzen

Go directly to Proof sequences for $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$

Deduction Systems: first try

For $\Sigma \subseteq$ **WFF** we could start to define $Ded(\Sigma)$ as follows:

Basis:
$\mathbf{T} \in Ded(\Sigma)$, $\Sigma \subseteq Ded(\Sigma)$

Closure:
If $\phi, \psi \in Ded(\Sigma)$ so also $(\phi \wedge \psi)$ and $(\psi \wedge \phi)$.
If $(\phi \wedge \psi) \in Ded(\Sigma)$ so also ϕ and ψ.

If $\phi, \psi \in Ded(\Sigma)$ so also $(\phi \to \psi)$ and $(\psi \to \phi)$.
If $\phi, (\phi \to \psi) \in Ded(\Sigma)$ so also ψ.

If $\phi \in$ **WFF**$, \psi \in Ded(\Sigma)$ so also $(\phi \vee \psi) \in Ded(\Sigma)$ and $(\psi \vee \phi) \in Ded(\Sigma)$.

If $\mathbf{F}, (\phi \to \mathbf{F}) \in Ded(\Sigma)$ so also $\phi \in Ded(\Sigma)$.
If $\phi \in$ **WFF**$, \mathbf{F} \in Ded(\Sigma)$ so also $\phi \in Ded(\Sigma)$.

If $\phi, \neg\phi \in Ded(\Sigma)$, so $\mathbf{F} \in Ded(\Sigma)$.

is this correct ? . **is this enough ?**

There will be many cases in the closure. We want to make our system **complete** but as simple as possible.

Go directly to Proof sequences for **WFF**$_{\{\to, \mathbf{F}\}}$

Deduction Systems: second try

For $\Sigma \subseteq$ **WFF** we could start to define $Ded(\Sigma)$ as follows:

Basis:
$\mathbf{T} \in Ded(\Sigma)$, $\Sigma \subseteq Ded(\Sigma)$

$(\phi \to (\psi \to \phi))$
$((\phi \to (\psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta)))$
$((\neg\psi \to \neg\phi) \to ((\neg\psi \to \phi) \to \psi))$

$((\phi \wedge \psi) \to \phi) \in Ded(\Sigma)$ and $((\phi \wedge \psi) \to \psi) \in Ded(\Sigma)$
$(\phi \to (\phi \vee \psi)) \in Ded(\Sigma)$ and $(\psi \to (\phi \vee \psi)) \in Ded(\Sigma)$

is this enough ?

Closure:

If $\mathbf{F}, (\phi \to \mathbf{F}) \in Ded(\Sigma)$ so also $\phi \in Ded(\Sigma)$.

is this enough ?

There will be many cases in the basis.
We want to make our system **complete** but as simple as possible.

Go directly to Proof sequences for $\mathbf{WFF}_{\{\to,\mathbf{F}\}}$

Homework

A sound system

For $\Sigma \subseteq$ **WFF** let $Ded_1(\Sigma)$ be defined by

Basis: $\Sigma \subseteq Ded_1(\Sigma)$

Closure:
If $\phi, \psi \in Ded_1(\Sigma)$ so also $(\phi \wedge \psi)$ and $(\psi \wedge \phi)$.
If $(\phi \wedge \psi) \in Ded_1(\Sigma)$ so also ϕ and ψ.

If $\phi, \psi \in Ded_1(\Sigma)$ so also $(\phi \to \psi)$ and $(\psi \to \phi)$.
If $\phi, (\phi \to \psi) \in Ded_1(\Sigma)$ so also ψ.

If $\phi \in$ **WFF**, $\psi \in Ded_1(\Sigma)$ so also $(\phi \vee \psi) \in Ded_1(\Sigma)$ and $(\psi \vee \phi) \in Ded_1(\Sigma)$.

If $\mathbf{F}, (\phi \to \mathbf{F}) \in Ded_1(\Sigma)$ so also $\phi \in Ded_1(\Sigma)$.
If $\phi \in$ **WFF**, $\mathbf{F} \in Ded_1(\Sigma)$ so also $\phi \in Ded_1(\Sigma)$.

If $\phi, \neg\phi \in Ded_1(\Sigma)$, so $\mathbf{F} \in Ded_1(\Sigma)$.

Show that $Ded_1(\Sigma) \subseteq Con(\Sigma)$ but $Con(\Sigma) \not\subseteq Ded_1(\Sigma)$.

Go directly to Proof sequences for **WFF**$_{\{\to,\mathbf{F}\}}$

Homework

A sound and complete system

For $\Sigma \subseteq$ **WFF** let $Ded_2(\Sigma)$ be defined by

Basis:
$\mathbf{T} \in Ded_2(\Sigma)$, $\Sigma \subseteq Ded_2(\Sigma)$, and all of the following are in $Ded_2(\Sigma)$:

$(\phi \to (\psi \to \phi))$,
$((\phi \to (\psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta)))$,
$((\neg\psi \to \neg\phi) \to ((\neg\psi \to \phi) \to \psi))$,

$((\phi \wedge \psi) \to \phi)$ and $((\phi \wedge \psi) \to \psi)$

$(\phi \to (\phi \vee \psi))$ and $(\psi \to (\phi \vee \psi))$

Closure:

If $\phi, (\phi \to \psi) \in Ded(\Sigma)$ so also $\psi \in Ded(\Sigma)$.

Show that $Ded_2(\Sigma) \subseteq Con(\Sigma)$.

$Ded_2(\Sigma)$ is also complete, i.e. $Con(\Sigma) \subseteq Ded_2(\Sigma)$.
You can show this later using the Completeness Theorem 25.

Go directly to Proof sequences for $\mathbf{WFF}_{\{\to,\mathbf{F}\}}$

Deduction Systems: Resolution

In the booklet you will also find

Resolution

used in AI, Logic Programming and Datalog.

It is defined for formulas in **CNF** only, and tests satisfiability, rather than logical consequence.

Resolution is more fit for machine implementations,
and in this sense it is more machine friendly.

Back to outline of Lecture 8-9

Proof Sequences $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$

To keep things simple we restrict ourselves to the case of propositional formulas in

$$\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}} \subseteq \mathbf{WFF}$$

Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ be a set of formulas.

We first shall define inductively the set $Ded(\Sigma)$
of formulas **deducible** from Σ.

A generating sequence of formulas

$$\phi_1, \phi_2, \ldots \phi_n = \phi$$

which shows that $\phi \in Ded(\Sigma)$ will be called a

deduction of ϕ from Σ
or a proof sequence for ϕ using Σ.

Why $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$?

The restriction to formulas in $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ is not really a restriction.

Proposition 1
Every formula $\phi \in \mathbf{WFF}$ is logically equivalent to a formula ψ such that

(i) $\psi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ and

(ii) ϕ and ψ have the same variables.

Proof:
Use that $\neg\phi$ is equivalent to $(\phi \rightarrow \mathbf{F})$ and
$(\phi \vee \psi)$ is equivalent to
$((\phi \rightarrow \mathbf{F}) \rightarrow \psi)$ **Q.E.D.**

The same holds for $\mathbf{WFF}_{\{\rightarrow,\neg\}}$ or $\mathbf{WFF}_{\{\wedge,\neg\}}$ or $\mathbf{WFF}_{\{\vee,\neg\}}$.

The Axioms for $\mathbf{F}, \rightarrow$

(after Lyndon)

Definition 2 (3.3.1)
*For every $\phi, \psi, \theta \in$ **WFF**$_{\{\rightarrow,\mathbf{F}\}}$ the following formulas are axioms.*

A1: $(\phi \rightarrow (\psi \rightarrow \phi))$

A2: $((\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta)))$

A3: $(((\phi \rightarrow \mathbf{F}) \rightarrow \mathbf{F}) \rightarrow \phi)$

We have seen already

Proposition 3 (3.2.3)
The axioms are logically valid (tautologies)

Skip variant

The Axioms for $\neg, \rightarrow$

(after Mendelson)

Definition 4 (in other courses)
*For every $\phi, \psi, \theta \in$ **WFF**$_{\{\rightarrow,\neg\}}$ the following formulas are axioms.*

AA1: $(\phi \rightarrow (\psi \rightarrow \phi))$

AA2: $((\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta)))$

AA3: $((\neg\psi \rightarrow \neg\phi) \rightarrow ((\neg\psi \rightarrow \phi) \rightarrow \psi))$

It is easy to see (as A1=AA1, A2=AA2)

Proposition 5
The axioms are logically valid (tautologies)

Back to outline of Lecture 8-9

Definition 6 (3.3.2)

Deducible formulas

Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ be a set of formulas.
We define inductively the set $Ded(\Sigma)$:

Basis:

(i) $\Sigma \subseteq Ded(\Sigma)$;

(ii) If $\phi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ is an **axiom**
then $\phi \in Ded(\Sigma)$.

Closure: (Modus ponens)

If $\phi \in Ded(\Sigma)$ and $(\phi \rightarrow \psi) \in Ded(\Sigma)$ then $\psi \in Ded(\Sigma)$.

Definition 7 (3.3.5)

Proof Sequence

To show that a formula $\phi \in Ded(\Sigma)$ one has to unwind the inductive definition of $Ded(\Sigma)$. Such an unwinding (generating sequence) will be called a

proof sequence

Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ be a set of formulas and $\phi_1, \ldots, \phi_n$ be formulas in $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$

We say that $\phi_1, \ldots, \phi_n$ is a **proof sequence over Σ** if for each $i \leq n$ either

(i) ϕ_i is an axiom or $\phi_i \in \Sigma$ or

(ii) (Modus ponens) there are $k, l < i$ such that $\phi_l = (\phi_k \rightarrow \phi_i)$

We write $\Sigma \vdash \phi$ if there is a proof sequence $\phi_1, \ldots, \phi_n$ over Σ such that $\phi_n = \phi$.

Proposition-Exercise 8 (3.3.6)

Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ be a set of formulas and $\phi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

Then $\phi \in Ded(\Sigma)$ iff $\Sigma \vdash \phi$.

Remark 9

To check whether $\phi \in Ded(\Sigma)$ or $\Sigma \vdash \phi$ may be hard:

We have to FIND the proof sequence
(and we don't know how long it is)

To check whether a sequence

$$\phi_1, \phi_2, \ldots, \phi_n$$

is a proof sequence over Σ is simple.

Recognizing axioms is done best for formulas in tree form $WFTF$.

Example 10

Let $\Sigma = \{((\phi \to \mathbf{F}) \to \mathbf{F}), (\phi \to \psi), (\psi \to \theta)\}$.
We show $\Sigma \vdash \theta$.

$(((\phi \to \mathbf{F}) \to \mathbf{F}) \to \phi)$	(1,A3)
$((\phi \to \mathbf{F}) \to \mathbf{F})$	(2,$\in \Sigma$)
ϕ	(3,MP(1,2))
$(\phi \to \psi)$	(4,$\in \Sigma$)
ψ	(5,MP(3,4))
$(\psi \to \theta)$	(6,$\in \Sigma$)
θ	(7,MP(5,6))

Example 11

Let $\Sigma = \{\phi, (\phi \to \mathbf{F})\}$. *We show for every* θ $\Sigma \vdash \theta$.

$$\phi \qquad (1, \in \Sigma)$$
$$(\phi \to \mathbf{F}) \qquad (2, \in \Sigma)$$
$$\mathbf{F} \qquad (3, \text{MP}(1,2))$$
$$(\mathbf{F} \to ((\theta \to \mathbf{F}) \to \mathbf{F})) \qquad (4, \text{A1}))$$

with $\phi = \mathbf{F}, \psi = (\theta \to \mathbf{F})$

$$((\theta \to \mathbf{F}) \to \mathbf{F}) \qquad (5, \text{MP}(3,4))$$
$$(((\theta \to \mathbf{F}) \to \mathbf{F}) \to \theta) \qquad (6, \text{A3})$$

with $\phi = \theta$

$$\theta \qquad (7, \text{MP}(6,5))$$

Soundness and Completeness, once more

$Ded(\Sigma)$ turns out to be an inductive definition of the logical consequences of Σ.

Soundness:
If $\phi \in Ded(\Sigma)$ then $\phi \in Con(\Sigma)$
or, equivalently
If $\Sigma \vdash \phi$ then $\Sigma \models \phi$

Completeness:
If $\phi \in Con(\Sigma)$ then $\phi \in Ded(\Sigma)$
or, equivalently
If $\Sigma \models \phi$ then $\Sigma \vdash \phi$

Proposition-Exercise 12 (3.3.3 and 3.3.7)

Soundness of $Ded(\Sigma)$

Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ be a set of formulas and $\phi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

If $\phi \in Ded(\Sigma)$ then $\Sigma \models \phi$.

Proof:

- Induction on the definition of $Ded(\Sigma)$.
- The axioms are tautologies (3.2.3)
- For Σ this is trivial.
- For Modus ponens we seen this in Lecture 7, slide 16, Proposition 14 (3.2.10).

Q.E.D.

Examples 13 (3.3.4 and 3.3.8)

Prove the following statements:

(i) $Ded(\emptyset)$ is a subset of the tautologies of $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.
...
If $\emptyset \vdash \phi$ then ϕ is a tautology.

(ii) $Ded(\{\mathbf{F}\}) = \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$
...
For every $\phi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ we have that $\mathbf{F} \vdash \phi$.

(iii) Let Σ be infinite and $\phi \in Ded(\Sigma)$.
Then there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\phi \in Ded(\Sigma_0)$.
...
Let Σ be infinite and $\Sigma \vdash \phi$ there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \phi$.

Definition 14 (3.3.9)

Consistency

We say that $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ is inconsistent if $\Sigma \vdash \mathbf{F}$.

If Σ is not inconsistent, we say that Σ is consistent.

Σ is consistent iff $\mathbf{F} \notin Ded(\Sigma)$ iff there is no proof sequence $\phi_1, \phi_2, \ldots, \phi_n$ over Σ with $\phi_n = \mathbf{F}$

Remark 15 (3.3.10)
By the soundness of proof sequences,
if Σ is inconsistent, then Σ is not satisfiable.

Proposition 16 (3.3.14)

Substitution in Proof Sequences

Proof sequences capture the essence of proofs and can be used for similar formulas in the following sense:

Let $\Sigma \vdash \phi$ and $s : Var \to$ **WFF** be a substitution.
Then

$$subst(\Sigma, s) = \{subst(\psi, s) : \psi \in \Sigma\} \vdash subst(\phi, s).$$

Proof:
If

$$\phi_1, \phi_2, \ldots \phi_n = \phi$$

is a proof sequence over Σ, so

$$subst(\phi_1, s), subst(\phi_2, s), \ldots subst(\phi_n, s) = subst(\phi, s)$$

is a proof sequence over $subst(\Sigma, s)$.

We have to verify this for axioms and Modus Ponens (exercise). **Q.E.D.**

Proposition 17 (3.3.11)

Manipulations of Proof Sequences

The following are useful manipulations of proof sequences.

Let $\Sigma_0 \subseteq \Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ and $\phi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

(i) If $\Sigma_0 \vdash \phi$ then $\Sigma \vdash \phi$;

(ii) If $\phi_1, \phi_2, \ldots, \phi_n$ is a proof sequence over Σ then for each $i \leq n$ we have that $\Sigma \vdash \phi_i$.

(iii) If $\Sigma \vdash \phi$ and $\Sigma \vdash (\phi \rightarrow \psi)$ then $\Sigma \vdash \psi$.

(iv) If $\Sigma \vdash (\phi \rightarrow (\theta \rightarrow \psi))$ and $\Sigma \vdash (\phi \rightarrow \theta)$ then $\Sigma \vdash (\phi \rightarrow \psi)$.

Proof of Proposition 17 (iv)

Proof: To show (iv) in proposition 17 it suffices to show

$$\{(\phi \to (\theta \to \psi)), (\phi \to \theta)\} \vdash (\phi \to \psi)$$

We proceed as follows:

$((\phi \to (\theta \to \psi)) \to ((\phi \to \theta) \to (\phi \to \psi)))$	(1,A2)
$(\phi \to (\theta \to \psi))$	(2,$\in \Sigma$)
$((\phi \to \theta) \to (\phi \to \psi))$	(3,MP(1,2))
$(\phi \to \theta)$	(4,$\in \Sigma$)
$(\phi \to \psi)$	(5,MP(4,3))

Q.E.D.

Theorem 18 (3.3.12)

Deduction Theorem

Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ be a set of formulas and $\phi, \psi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

$\Sigma \vdash (\phi \rightarrow \psi)$ iff $\Sigma \cup \{\phi\} \vdash \psi$.

The proof is part of the Tirgulim

Skip the proof of the Deduction Theorem

Proof of Theorem 18, I

(i) Assume $\Sigma \vdash (\phi \to \psi)$.

We have to prove that $\Sigma \cup \{\phi\} \vdash \psi$.

By proposition 17 (i) we have
$\Sigma \cup \{\phi\} \vdash (\phi \to \psi)$ and,
using modus ponens,
$\Sigma \cup \{\phi\} \vdash \psi$.

Proof of Theorem 18, II

(ii) Assume $\Sigma \cup \{\phi\} \vdash \psi$.

We have to show that $\Sigma \vdash (\phi \to \psi)$.

Equivalently, we can show that
$(\phi \to \psi) \in Ded(\Sigma)$.

Let $K \subseteq Ded(\Sigma \cup \{\phi\})$ be the set of formulas ψ such that $\Sigma \vdash (\phi \to \psi)$.

We show by induction that
$K = Ded(\Sigma \cup \{\phi\})$.

Proof of Theorem 18, III

Basis:
$\Sigma \subseteq K$ and all the axioms are in K (Exercise).

In the case that $\psi = \phi$ we use that $\emptyset \vdash (\phi \to \phi)$.

Closure:
Assume θ and $(\theta \to \psi)$ are in K.

We have to show that $\psi \in K$.

By assumption $\Sigma \vdash (\phi \to \theta)$ and $\Sigma \vdash (\phi \to (\theta \to \psi))$.

Therefore, by proposition 17 (iv), $\Sigma \vdash (\phi \to \psi)$. Q.E.D.

Back to outline of Lecture 8-9

Theorem 19 (3.3.13)

Dychotomy Theorem

Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ be a set of formulas and $\phi, \psi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

If both $\Sigma \cup \{\phi\} \vdash \psi$ and
$\Sigma \cup \{(\phi \rightarrow \mathbf{F})\} \vdash \psi$.

Then $\Sigma \vdash \psi$.

Proof of the Dychotomy Theorem 19, I

Use the Deduction Theorem to prove

$$\{(\phi \to \psi), (\psi \to \theta)\} \vdash (\phi \to \theta)$$

Then use this to prove the following three tautologies:

$$((\phi \to \psi) \to ((\psi \to \mathbf{F}) \to (\phi \to \mathbf{F})))$$

$$(((\phi \to \mathbf{F}) \to \psi) \to ((\psi \to \mathbf{F}) \to ((\phi \to \mathbf{F}) \to \mathbf{F})))$$

$$(((\psi \to \mathbf{F}) \to ((\phi \to \mathbf{F}) \to \mathbf{F})) \to (((\psi \to \mathbf{F}) \to (\phi \to \mathbf{F})) \to \psi))$$

Then use these three tautologies and Modus Ponens to prove the Dychotomy Theorem. Q.E.D.

Example 20

We show $\emptyset \vdash (\phi \to \phi)$:

With Deduction theorem it suffices to show $\phi \vdash \phi$ which is trivial.

*****

Without Deduction Theorem we proceed as follows:

$$((\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi))) \qquad (1,A2)$$

with $\phi = \theta, \psi = (\theta \to \theta)$

$$(\phi \to ((\phi \to \phi) \to \phi)) \qquad (2,A1)$$

with $\psi = (\phi \to \phi)$

$$((\phi \to (\phi \to \phi)) \to (\phi \to \phi)) \qquad (3,MP(1,2))$$

$$((\phi \to (\phi \to \phi)) \qquad (4,A1)$$

$$(\phi \to \phi) \qquad (5,MP(3,4))$$

Q.E.D.

Back to outline of Lecture 8-9

Example 21 (Double Negation)
We show $\phi \vdash ((\phi \to \mathbf{F}) \to \mathbf{F})$.

With Deduction Theorem it suffices to show

$$\{\phi, (\phi \to \mathbf{F})\} \vdash \mathbf{F}$$

This is achieved using once Modus Ponens.

*****

Exercise:
Prove it without the Deduction Theorem.

Example 22 (Contrapositive)
We show $(\phi \to \psi) \vdash ((\psi \to \mathbf{F}) \to (\phi \to \mathbf{F}))$.

With Deduction Theorem it suffices to show

$$\{(\phi \to \psi), (\psi \to \mathbf{F}), \phi\} \vdash \mathbf{F}$$

But this is achieved using twice Modus Ponens.

*****

Exercise:
Prove it without the Deduction Theorem.

Example 23 (Disjunctions)

We want to show $\phi \vdash (\psi \vee \phi)$

We replace $\vee$ using $\mathbf{F}$ and $\rightarrow$:

$$\phi \vdash ((\psi \rightarrow \mathbf{F}) \rightarrow \phi)$$

$(\phi \rightarrow ((\psi \rightarrow \mathbf{F}) \rightarrow \phi))$	(1,A1)
ϕ	(2,$\in \Sigma$)
$((\psi \rightarrow \mathbf{F}) \rightarrow \phi)$	(3,MP(1,2))

Q.E.D.

We do not use the Deduction Theorem here.

Example 24 (Conjunction)

We want to show $\theta = (\phi \wedge \psi) \vdash \psi$

We replace $\wedge$ using $\mathbf{F}$ and $\rightarrow$:

$$\theta = \neg(\neg\phi \vee \neg\psi) = \neg(\neg\neg\phi \rightarrow \neg\psi) = ((((\phi \rightarrow \mathbf{F}) \rightarrow \mathbf{F}) \rightarrow (\psi \rightarrow \mathbf{F})) \rightarrow \mathbf{F})$$

Using the Deduction Theorem it suffices to prove

$$\vdash (\theta \rightarrow \psi)$$

Using the Contrapositive it suffices to prove

$$\vdash ((\psi \rightarrow \mathbf{F}) \rightarrow (\theta \rightarrow \mathbf{F}))$$

Using the Contrapositive and the Deduction Theorem once more we need

$$\{(\psi \rightarrow \mathbf{F}), \theta\} \vdash \mathbf{F}$$

For this we look at A2 in the form

$$((\psi \rightarrow \mathbf{F}) \rightarrow (((\phi \rightarrow \mathbf{F}) \rightarrow \mathbf{F}) \rightarrow (\psi \rightarrow \mathbf{F})))$$

and use Modus Ponens twice. Q.E.D.

Homework

Prove all of the following tautologies using proof sequences:

T

$(\phi \to (\psi \to \phi))$
$((\phi \to (\psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta)))$
$((\neg\psi \to \neg\phi) \to ((\neg\psi \to \phi) \to \psi))$

$((\phi \wedge \psi) \to \phi) \in Ded(\Sigma)$
$((\phi \wedge \psi) \to \psi) \in Ded(\Sigma)$

$(\phi \to (\phi \vee \psi)) \in Ded(\Sigma)$

$(\psi \to (\phi \vee \psi)) \in Ded(\Sigma)$

Is this enough to prove completenss?

Homework

To pass between $\mathbf{WFF}_{\{\neg,\to\}}$ and $\mathbf{WFF}_{\{\mathbf{F},\to\}}$ we use the definition of $\neg$ and $\mathbf{F}$:

- $\neg\phi$ is defined by $(\phi \to \mathbf{F})$ and
- $\mathbf{F}$ is defined by $\neg(p_0 \to p_0)$

(i) Prove the following tautologies using proof sequences in $\mathbf{WFF}_{\{\mathbf{F},\to\}}$:
$(\mathbf{F} \to ((p_0 \to p_0) \to \mathbf{F}))$ and $(((p_0 \to p_0) \to \mathbf{F}) \to \mathbf{F})$

(ii) Prove the following tautologies using proof sequences in $\mathbf{WFF}_{\{\neg,\to\}}$:
$(\neg\phi \to (\phi \to \neg(p_0 \to p_0)))$ and $((\phi \to \neg(p_0 \to p_0)) \to \neg\phi)$

Back to outline of Lecture 8-9

Completeness Theorem for Deductions

The following shows that the method of proof sequences is sufficiently powerful to obtain all tautologies, or, more generally, all logical consequences of a given set of formulas.

Theorem 25 (3.3.15)
Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ *and* $\phi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

$$\Sigma \models \phi \text{ iff } \Sigma \vdash \phi.$$

Proof: We have seen the soundness: If $\Sigma \vdash \phi$ then $\Sigma \models \phi$.

The proof that if $\Sigma \models \phi$ then $\Sigma \vdash \phi$ will be given in the next lecture. Q.E.D.

Corollary 26 (3.3.21)
Compactness Theorem

Let $\Sigma \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ be an infinite set of formulas.

Σ is satisfiable iff every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable.

Proof of the Compactness Theorem

Proof:
By the completeness theorem above Σ is satisfiable iff Σ is consistent.

Clearly, if Σ is consistent, so is every finite subset $\Sigma_0 \subseteq \Sigma$.

Conversely, assume Σ is inconsistent.
Therefore there is a proof sequence

$$\psi_1, \ldots, \psi_k, \theta_1, \ldots \theta_m, \mathbf{F}$$

with exactly the $\psi_i \in \Sigma$,
which shows the inconsistency.

Therefore,

$$\{\psi_1, \ldots, \psi_k\} \subseteq \Sigma$$

is a finite inconsistent subset. **Q.E.D.**

Example 27

Colorability of graphs

Let $red_i, green_i, blue_i, white_i,$ and $e_{i,j}$ be propositional variables for $i, j \in \mathbb{N}$.

Let $G \subseteq \mathbb{N}^2$.

- $GRAPH = \{e_{i,j} : (i,j) \in G\}$ says
  there is an edge between i and j.

- $COLOR_i = (red_i \vee green_i \vee blue_i \vee white_i)$ says
  each i has one of the four colors.

- $CR_i = (red_i \rightarrow (\neg blue_i \wedge \neg green_i \wedge \neg white_i))$ says
  if i is red, then it has no other color.
  Similarly for CG_i, CB_i and CW_i.

- $NR_{i,j} = ((red_i \wedge e_{i,j}) \rightarrow \neg red_j)$ says
  if i is red and j is a neighbor of i, then j is not red.
  Similarly for NG_i, NB_i and NW_i.

Example 28

Colorability of graphs, contd

Now we put

- $COLOR = \{COLOR_i : i \in \mathbb{N}\}$
- $C = \{(CR_i \wedge CB_i \wedge CW_i \wedge CG_i) : i \in \mathbb{N}\}$
- $N = \{(NR_i \wedge NB_i \wedge NW_i \wedge NG_i) : i \in \mathbb{N}\}$
- $\Sigma_G = GRAPH \cup COLOR \cup C \cup N$

For a fixed G Σ_G is satisfiable iff
$GRAPH$ can be colored with four colors.

Compactness now says:
If every finite subgraph of $GRAPH$ can be colored with four colors,
so can $GRAPH$.

Proof of the Completeness Theorem

To prove the Completeness theorem we observe first that it suffices to prove

$$\Sigma \text{ is satisfiable iff } \Sigma \text{ is consistent.}$$

Proof: Assume $\Sigma \models \phi$.
This is equivalent to $\Sigma_1 = \Sigma \cup \{\neg\phi\} \models \mathbf{F}$
i.e. Σ_1 is not satisfiable.
But then Σ_1 is inconsistent, hence
$\Sigma_1 = \Sigma \cup \{(\phi \to \mathbf{F})\} \vdash \mathbf{F}$ and,
using the Deduction Theorem,
$\Sigma \vdash ((\phi \to \mathbf{F}) \to \mathbf{F})$ and, using A3, $\Sigma \vdash \phi$.
...
The other direction is soundness. Q.E.D.

How we prove the Completeness Theorem

We reduce it to showing that
if a set $\Sigma \subseteq WFF_{\{\rightarrow,\mathbf{F}\}}$ is consistent,
then it has a satisfiable assignment z.

Σ may have many assignements,
and they may be hard to describe.

So we reduce the choices:
Finding a conistent $\Sigma' \supseteq \Sigma$ will have fewer satisfying assignments
(if any at all).

We will construct $\Sigma_\omega \supseteq \Sigma$ with at most one satisfying assignment.

Then we show that indeed it has
exactly one satisfying assignment:

$$z(p_i) = 1 \text{ iff } p_i \in \Sigma_\omega$$

How to find proof sequences

The above proof of the completeness theorem
is mathematically very elegant,
but hides the construction of proof sequence,
whose very existence we show.

This proof method was independently suggested by
L.Henkin, G. Hasenjaeger and J.Hintikka
(all in 1949).

The actual **construction** of proof sequences was shown by
P. Bernays (1918), E. Post (1921) and L. Kalmar (1934).

It is the topic of courses in

Automated Theorem Proving or in Textbooks on Logic prior to 1960.

Our proof adapts easily to First Order Logic.

Skip detailed proof of the Completeness Theorem
Back to outline of Lecture 8-9

Maximally consistent sets

We use the notions

consistent and **inconsistent**

also for sets $\Sigma \subseteq \mathbf{WFF}_{\{\mathbf{F},\rightarrow\}}$.

Definition 29 (3.3.16)
A set Σ *of* $\mathbf{WFF}\{\rightarrow,\mathbf{F}\}$ *is* maximally consistent
if it is consistent,
and for every $\phi \in \mathbf{WFF}\{\rightarrow,\mathbf{F}\}$
either $\Sigma \vdash \phi$ *or* $\Sigma \vdash (\phi \rightarrow \mathbf{F})$.

Maximally consistent extensions

Lemma 30 (3.3.17)
Let $\Sigma \subseteq \mathbf{WFF}\{\rightarrow, \mathbf{F}\}$ *be consistent.*

Then there is a maximally consistent Σ^*
such that $\Sigma \subseteq \Sigma^*$.

The proof is in **four stages**.

Proof: [Stage (i)]

Assume now, that Σ is consistent.

Let $\{\phi_i : i \in \mathbb{N}\}$ be an enumeration of the formulas in $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

We define in stages a set $\Sigma_\omega \subseteq \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ in the following way:

- $\Sigma_0 = \Sigma$;
- $\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\phi_n\} \text{ if } \Sigma_n \cup \{\phi_n\} \text{ is consistent, and} \\ \Sigma_n \cup \{(\phi_n \rightarrow \mathbf{F})\} \text{ otherwise.} \end{cases}$
- $\Sigma_\omega = \bigcup\{\Sigma_i : i \in \mathbb{N}\}$.

Q.E.D.

Proof: [Stage (ii)]

For every $i \in \mathbb{N}$ we use the Dychotomy Theorem to show that Σ_{i+1} is consistent.

Assume Σ_i is consistent, but
$\Sigma_i \cup \{\phi_i\} \vdash \mathbf{F}$
and
$\Sigma_i \cup \{(\phi_i \to \mathbf{F})\} \vdash \mathbf{F}$.

Using the Dychtomy Theorem we get that $\Sigma_i \vdash \mathbf{F}$ hence, Σ_i is inconsistent, a contradiction. **Q.E.D.**

Proof: [Stage (iii)]

Σ_ω is consistent.

For otherwise, there are formulas $\{\psi_1, \ldots, \psi_k\}$ such that

$$\{\psi_1, \ldots, \psi_k\} \vdash \mathbf{F}.$$

So there is $m \in \mathbb{N}$ such that

$$\psi_1, \ldots, \psi_k, \mathbf{F} \in \Sigma_m$$

and therefore, Σ_m is inconsistent, contrary to Stage (ii). **Q.E.D.**

Proof: [Stage (iv)]

Σ_ω is a maximally consistent extensions of Σ,
i.e. for every $\psi \in \mathbf{WFF}_{\{\to,\mathbf{F}\}}$
either $\Sigma_\omega \vdash \psi$ or $\Sigma_\omega \vdash (\psi \to \mathbf{F})$.

As $\{\phi_i : i \in \mathbb{N}\}$ is an enumeration
of the formulas in $\mathbf{WFF}_{\{\to,\mathbf{F}\}}$,
$\psi = \phi_n$ for some $n \in \mathbb{N}$.

Therefore $\psi \in \Sigma_n$ or $(\psi \to \mathbf{F}) \in \Sigma_n$.

Now we put $\Sigma^* = \Sigma_\omega$, which completes the proof. **Q.E.D.**

Remark 31 (3.3.18)

Note that different enumerations of $\mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$ give different sets Σ_ω for the same Σ.

In general, for countable Σ, there are $2^{\mathbb{N}}$ maximally consistent extensions.

Exercise:
Prove this.

Assignment for maximally consistent set

Lemma 32 (3.3.19)
Let $\Sigma \subseteq \mathbf{WFF}\{\rightarrow, \mathbf{F}\}$ *be maximally consistent. Then there is an assignment* $z : Var \rightarrow \{0, 1\}$ *such that* $M_{PL}(\Sigma, z) = 1$.

In other words, Σ *is satisfiable.*

Proof: [Stage (i)]

We first define a propositional assignment

$$z : Var \to \{0, 1\}$$

for Σ in the following way:

$$z(p_i) = 1 \text{ iff } p_i \in \Sigma$$

As Σ is maximally consistent,
this is well defined. **Q.E.D.**

Proof:[Stage (ii)]

All we need to show now, is that
$M_{PL}(\psi, z) = 1$ iff $\Sigma \vdash \psi$.

This we do by induction for every $\psi \in \mathbf{WFF}_{\{\rightarrow,\mathbf{F}\}}$.

Basis:
For ψ atomic this follows from the definition of z.

Closure: Assume we have shown it for ψ_1, and that ψ_2, and $\psi = (\psi_1 \rightarrow \psi_2)$.

From left to right:

$M_{PL}(\psi, z) = 1$ iff
$M_{PL}((\psi_1 \rightarrow \psi_2), z) = TT_{\rightarrow}(M_{PL}(\psi_1, z), M_{PL}(\psi_2, z)) = 1$.

By the induction hypothesis $M_{PL}(\psi_1, z) = 1$ iff $\Sigma \vdash \psi_1$.

So assume, for contradiction, that
$M_{PL}(\psi_1, z) = 1$, $M_{PL}(\psi_2, z) = 0$ and $\Sigma \vdash \psi$.

Then $\Sigma \vdash \psi_1$, $\Sigma \vdash (\psi_2 \rightarrow \mathbf{F})$, and also $\Sigma \vdash \psi$,

which contradicts the consistency of Σ.

Proof:[**Stage (ii)**] continued

From right to left:

$M_{PL}(\psi, z) = 0$ iff
$M_{PL}((\psi_1 \to \psi_2), z) = M_{\to}(M_{PL}(\psi_1, z), M_{PL}(\psi_2, z)) = 0$.

So assume, for contradiction, that
$M_{PL}(\psi_2, z) = 1$ and $\Sigma \not\vdash \psi$.

Then, by maximal consistency, $\Sigma \vdash (\psi \to \mathbf{F})$, hence
$\Sigma \vdash ((\psi_1 \to \psi_2) \to \mathbf{F})$.

By induction hypothesis, $\Sigma \vdash \psi_2$ iff $M_{PL}(\psi_2, z) = 1$
hence $\Sigma \vdash \psi_2$.

Using Axiom $mathbf A1$, and the Deduction Theorem, we get that

$$((\psi_1 \to \psi_2) \to \mathbf{F}), \psi_2 \vdash \mathbf{F}$$

hence Σ is inconsistent.

Conclusion:

$M_{PL}(\Sigma, z) = 1$, and as $\Sigma \subseteq \Sigma$, $M_{PL}(\Sigma, z) = 1$ which shows that Σ is satisfiable.
Q.E.D.

End of the proof of the Completeness Theorem

Proof: The proof is in several stages.

(i) We have observed already that it suffices to prove that

If Σ is consistent then Σ is satisfiable.

For this we used the Deduction and the Dychotomy Theorems.

(ii) By the soundness of proof sequences,

if Σ is satisfiable, then Σ is consistent.

(iii) Assume now, that Σ is consistent.

So there is Σ^* maximally consistent with $\Sigma \subseteq \Sigma^*$ by lemma 30.
But Σ^* is satisfiable by lemma 32. **Q.E.D.**

Back to outline of Lecture 8-9

Applications of Compactness: **Definability**

Definition 33 (3.4.4)
We look now at sets K of assignments which are defined

*by sets of formulas $\Sigma subseteq$***WFF**$\{\rightarrow, \mathbf{F}\}$.

(i) Let Σ be a set of formulas in **WFF**$\{\rightarrow, \mathbf{F}\}$.
We define

$$\text{Ass}(\Sigma) = \{z \in \text{Ass} : M_{PL}(\Sigma, z) = 1\}.$$

(ii) Let K be a subset of Ass.
We say that K is **(set)-definable in WFF**$\{\rightarrow, \mathbf{F}\}$
if there is a set $\Sigma \subseteq$ **WFF**$\{\rightarrow, \mathbf{F}\}$ such that $K = \text{Ass}(\Sigma)$.

(iii) A set K of truth assignments is **finitely definable**
if there is a **finite** Σ such that $K = \text{Ass}(\Sigma)$.

Examples 34 (3.4.5)

(o) $\mathrm{Ass}(\mathbf{T}) = \mathrm{Ass}$,
the set of all truth assignments.

(i) Let K_1 be the set of
truth assignments z such that $z(p_0) = 1$.

Then $K_1 = \mathrm{Ass}(\{p_0\})$, and therefore
is finitely definable.

(ii) Let K_2 be the set of
truth assignments z such that
$z(p_i) = z(p_{2i})$ for every $i \in \mathbb{N}$.

Then $K_2 = \mathrm{Ass}(\Sigma_2)$, for

$$\Sigma_2 = \{((p_i \to p_{2i}) \wedge (p_{2i} \to p_i)) : i \in \mathbb{N}\},$$

and therefore is set-definable.

Examples 35 (3.4.5, contd)

(iii) If $K = \text{Ass}(\Sigma)$ and $K' = \text{Ass}(\Sigma')$,
then $K \cap K'$ is definable with

$$K \cap K' = Ass(\Sigma \cup \Sigma').$$

(iv) If $K = \text{Ass}(\Sigma)$ and $K' = \text{Ass}(\Sigma')$,
then $K \cup K'$ is definable with

$$K \cup K' = Ass(\{\phi \vee \psi : \phi \in \Sigma, \psi \in \Sigma'\})$$

(v) If $\Sigma \subseteq \Sigma'$ then $\text{Ass}(\Sigma') \subseteq \text{Ass}(\Sigma)$.

Examples 36 (3.4.5, contd)

Exercises

(vi) Let $K = \text{Ass}(\Sigma)$ and $K' = \text{Ass}(\Sigma')$, what can you say about $\text{Ass}(\Sigma \cap \Sigma')$?

Exercise: Study several special cases.

(vii) Let

$$K_{\mathbf{T}} = \{z \in Ass : z(p_i) = 1, i \in \mathbb{N}\}$$

Note that $K_{\mathbf{T}}$ has exactly one element
and K_T is definable by $\Sigma_T = \{p_i : i \in \mathbb{N}\}$.

Exercise:
Show that $K_{\mathbf{T}}$ is not definable
by any finite subset $\Sigma \subseteq \Sigma_T$,
hence is not finitely definable.

Hint: In such a Σ there are only finitely many variables.

Examples 37 (3.4.5, contd)

Exercises

(viii) If $K = \mathsf{Ass}(\Sigma)$ for some finite Σ.

Then the complement
$K' = \mathsf{Ass} - K$
is also definable.

Exercise:
Describe Σ' which defines K'.

Main Theorem on Definability in Propositional Logic

Theorem 38 (proposition 3.4.6)

(i) Let $K \subseteq Ass$ be definable
and $\mathsf{Ass} - K$ be also definable, i.e.
$K = \mathsf{Ass}(\Sigma)$ and $\mathsf{Ass} - K = \mathsf{Ass}(\Sigma')$.

Then there is a finite subset $\Sigma_0 \subseteq \Sigma$
such that $K = \mathsf{Ass}(\Sigma_0)$.

(ii) Let K be a set of truth assignments.

K is finitely definable iff
both K and its complement are definable.

Proof of Theorem 38

(i) By assumption and example 34 (iii) above

$$K \cap K' = \emptyset = \mathsf{Ass}(\Sigma \cup \Sigma')$$

Therefore $\Sigma \cup \Sigma'$ is not satisfiable.

By the Compactness Theorem there is a finite subset

$$\Sigma_1 \subseteq \Sigma \cup \Sigma'$$

which is not satisfiable.

Let $\Sigma_0 = \Sigma \cap \Sigma_1$.
By example 35 (vi) above, we have $K \subseteq \mathsf{Ass}(\Sigma_0)$.

To show that $\mathsf{Ass}(\Sigma_0) \subseteq K$ it suffices to show, that $\mathsf{Ass}(\Sigma_0) \cap (\mathsf{Ass} - K) = \emptyset$.

But, as Σ_1 is not satisfiable, neither is $\Sigma_0 \cup \Sigma'$,
because $\Sigma_1 \subseteq \Sigma_0 \cup \Sigma'$.

(ii) follows from (i) and example 35 (viii) above. Q.E.D.

Exercises 39 (3.4.7)

Showing non-definability

(i) Show that $\mathsf{Ass} - K_{\mathbf{T}}$ is not definable.

(ii) Let K_{even} be the set of truth assignments z such that $z(p_{2i}) = 1$ for every $i \in \mathbb{N}$.

Show that K_{even} is not finitely definable.

(iii) Let $K_{fin-true}$ be the set of assignements z with finitely many i's such that $z(p_i) = 1$.

Show that $K_{fin-true}$ is not (set)-definable.

Homework

(i) Let $K \subseteq Ass$ and $z \notin K$.
Assume Σ defines K.
Let

$$\Sigma_1 = \{p_i : z(p_i) = 1\} \cup \{\neg p_i : z(p_i) = 0\}$$

Then $\Sigma \cup \Sigma_1$ is not satisfiable.

(ii) Let K be the set of assignments z such that for exactly one $i \in \mathbb{N}$ $z(p_i) = 1$.

Use (i) and compactness to show that neither K nor its complement $Ass \setminus K$ are definable.

Skip Finite Support and go to What Else?

Definition 40 (3.4.8)

Dependence on variables

Let K be a set of truth assignments.
Let D be a set of propositional variables.

We say that K
does depend on the variables $p_i \in D$
if there are truth assignments z and z'
which only differ for $p_i \in D$,
i.e. with $z(p_j) = z'(p_j)$ for every $p_j \notin D$,
such that $z \in K$ and $z' \notin K$.

Definition 41 (3.4.8)

Finite Support

Let K be a set of truth assignments.
Let S be a set of propositional variables.

(i) S is a support for K
if for any two assignments z and z'
which coincide on S,
i.e. for every $p_i \in S$ $z(p_i) = z'(p_i)$,
$z \in K$ iff $z' \in K$.

(ii) We say that K has finite support
if there is a finite S
which is a support for K.

Example 42

Two critical examples

Let K_1 be all the assignments z with $z(p_i) = 1$ for finitely many variables.

Let K_2 be all the assignments z with $z(p_{2i}) = 1$ for all $i \in \mathbb{N}$ and $z(p_{2i+1}) = 1$ finitely many $i \in \mathbb{N}$.

(i) The set of all variables is a support for K_1 and K_2.

(ii) A proper subset $S \subset \mathbb{N}$ is a support for K_1 iff $\mathbb{N} \setminus S$ is finite.

(iii) **Exercise:** Which proper subsets $S \subset \mathbb{N}$ are a support for K_2?

(iv) K_1 does not depend on any proper subset $D \subset \mathbb{N}$.

(v) K_2 does depend on each single variable p_{2i} (does depend on $\{p_{2i}\}$).

Homework

Let S and S' be supports for K
and D a set of variables.

Prove or disprove the following:

(i) If $D \subseteq Var \setminus S$ then K does not depend on D.

(ii) If K depends on D then $D \subseteq S$.

(iii) If $D \subseteq S$ then K depends on D.

(iv) $S \cap S'$ is a support for K.

(v) $S \cup S'$ is a support for K.

(vi) $S = \{p_i \in Var : K \text{ depends on } \{p_i\}\}$

(vii) $S = \bigcup\{D \subseteq Var : K \text{ depends on } D\}$

Examples 43 (3.4.9)

(i) If K is finitely definable,
then K has finite support.

(ii) K_{even} does not have finite support.

(iii) If K is itself finite,
it does not have finite support.

Theorem 44 (proposition 3.4.10)

Finite Support Theorem

Let K be a set of truth assignments.
K is finitely definable iff
K has finite support.

Proof: By (i) in the example above,
if K is finitely definable,
then K has finite support.

So assume, K has finite support.

Then it suffices to describe
the truth table describing K
on the variables of Support(K). **Q.E.D.**

Back to outline of Lecture 8-9

SAT: The most important open problem in Propositinal Logic

Input: Given a finite set $\Sigma \in CNF$.

Question: Is Σ satsifiable.

Find **efficient** algorithms for SAT.

Propositional Modal Logic

Propositional Logic analyses the words

and, or, not, if ... then

Modal Propositional Logic analyses additionally the words

possibly, necessarily

We study this in the course **Logic II**.

Propositional Temporal Logic

Propositional Logic analyses the words

and, or, not, if ... then

Temporal Propositional Logic analyses additionally the words

next time, previous time
some time in the future,
some time in the past,
always in the future,
always in the past,
until, since

We study this in the course **Logic II**.

Where do we use Propositional Logics

Propositional Logic (and its Modal and Temporal versions) are used in

- **Hardware** specification and verification.
- **Software** specification and verification.
- **Automated reasoning** and **Artificial Intelligence (AI)**.
- **Databases**.

Now we are ready for

Predicate Logic aka First Order Logic **FOL**

- Predicate Logic for Arithmetics
- Semantics of **FOL**
- Syntax of **FOL**
- Learning to read and understand formulas of **FOL**
- Handling quantifiers
- Expressive power of **FOL**
- Stating Gödel's Completeness Theorem

Outlook: Predicate aka First Order Logic **FOL**

FOL for the natural numbers with their arithmetic operations

- Inductive definition of the syntax
- Inductive definition of the meaning function
- Reading, writing and understanding **FOL**-formulas

FOL-structures as interpretation of vocabularies τ

- What is a **FOL**-vocabulary?
- Inductive definition of the syntax and semantics for **FOL**(τ)
- Proof systems for **FOL**.
- Definability and non-definability in **FOL**