Metafinite Model Theory

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joint work with Yuri Gurevich
(once upon a time . . .)

Finite model theory versus computability

Fundamental difference between logic and classical algorithmic models:

Logic preserves symmetries at every stage of the evaluation of a formula, or of an iterative process, whereas algorithms may break symmetries, for instance by explicit choices, or sequential processing of input elements along an ordering that is not inherent to the input structure, but just to its representation.

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A lot of the interesting questions, but also the technical difficulties, in finite model theory come from this mismatch between logic and classical computational devices. This mismatch only arises when we deal with abstract unordered structures. It disappears when we work with numbers, strings or other ordered objects.

But many interesting objects are hybrid. They consist of structures (with symmetries) and numbers.

Metafinite Model Theory: Motivation

Extend the approach and methods of finite model theory to a richer setting, which combines finite structures with objects from infinite structured domains, such as natural, real, or complex numbers with the common arithmetic operations.

Nevertheless these extension should preserve the spirit, objectives and methods of finite model theory, and the connections with challenges from various branches of computer science. Infinity should not manifest itself too obtrusively, deviating our attention to phenomena that are pertinent to infinite structures only.

Metafinite structures include weighted graphs, databases with numerical domains and aggregate operations, structures with probabilistic information, and so on.

Metafinite structures

A metafinite structure is a triple $\mathfrak{D} = (\mathfrak{A}, \mathfrak{R}, W)$ consisting of

(1) a finite structure $\mathfrak{A}$, for instance a graph, called the primary part of $\mathfrak{D}$;

(2) a (typically infinite) structure $\mathfrak{R}$, for instance the field of real numbers, possibly equipped also with multiset operations Π, mapping finite multisets over $\mathfrak{R}$ to elements of $\mathfrak{R}$.

(3) a finite set W of functions that map tuples in $\mathfrak{A}$ to elements of $\mathfrak{R}$.

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The role of multiset operations: Extend associative and commutative operations on R, such as $+$ and $\cdot$, to tuples of unbounded length. Any term $t(\overline{x})$ that defines on $\mathfrak{D}$ a function $t^{\mathfrak{D}} : A^k \to R$ gives rise to a multiset $t^{\mathfrak{D}}(A^k) = \{\{t^{\mathfrak{D}}(\overline{a}) : \overline{a} \in A^k\}\}$. By applying multiset operations to such terms we can, for instance, define sum and products with an unbounded number of arguments.

Metafinite structures: Examples

Arithmetical Structures: Metafinite structures $\mathfrak{D} = (\mathfrak{A}, \mathfrak{N}, W)$ where $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, <, \dots,)$ with multiset operations $\max, \min, \sum, \prod, \dots$ We require that all operation in $\mathfrak{N}$ can be evaluated in polynomial time.

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$\mathbb{R}$-structures: Metafinite structures where the secondary part is $\mathfrak{R} = (\mathbb{R}, +, -, \cdot, /, \leq, (c_r)_{r \in \mathbb{R}})$, so that every rational function can be written as a term. $\mathbb{R}$-structures have been used to develop a descriptive complexity theory related to the BSS-model of computation over the real numbers.

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Metafinite algebras: In principle we can always reduces the primary part of a metafinite structure to a naked finite set A, and push all the data into the weight functions from A to $\mathfrak{R}$. Important examples include elements of vector spaces over an infinite field.

Logics for metafinite structures

The common logics of finite model theory (FO, LFP, ...) extend to logics for reasoning about metafinite structures $\mathfrak{D} = (\mathfrak{A}, \mathfrak{R}, W)$. Such a logic has:

- point terms, defining functions $f : A^k \to A$
- weight terms, defining functions $w : A^k \to R$, and
- formulae, defining relations $R \subseteq A^k$.

The rules for building terms and formulae are standard, with the proviso that variables range over the primary part only. In addition, we have:

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Terms with multiset operations: If $F(\bar{x}, \bar{y})$ is a weight term, $\varphi(\bar{x}, \bar{y})$ a formula, and Γ is a multiset operation of $\mathfrak{R}$, then $\Gamma_{\bar{x}}(F(\bar{x}, \bar{y}) \ : \ \varphi)$ is a weight term with free variables $\bar{y}$. For any assignment $\bar{y} \mapsto \bar{b}$ in a structure $\mathfrak{D}$, this term takes the value $\Gamma(\{\{F^{\mathfrak{D}}(\bar{a}, \bar{b}) : \bar{a} \in A^k \text{ such that } \mathfrak{D} \models \varphi(\bar{a}, \bar{b})\}\})$.

Definability with multiset operations

Counting elements. In the presence of multiset operations, such as $\sum$, counting is definable by $\#_x[\varphi] := \sum_x \chi[\varphi]$.

Binary representation. Let $\mathfrak{A} = (\{0,\dots,n-1\},<,P)$. We can view P as the binary representation of a natural number $m(P) < 2^n$, which is definable by

$$\sum_x \Big(\chi[Px] \cdot \prod_y (2 : y < x)\Big)$$

Multiset operations are the basis of an adequate logical theory for the definability of numerical invariants of graphs and other finite structures, and for aggregate operations in relational databases.

Generalized Spectra and Fagin's Theorem

The spectrum of a first-order sentence is the set of cardinalities of its finite models.

$$\text{spectrum}(\psi) := \{n < \omega : \psi \text{ has a model with } n \text{ elements}\}.$$

A classical problem of mathematical logic: characterize the class of spectra.

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Connection with existential second-order logic Σ^1_1: The spectrum of a first-order sentence ψ with relations symbols $R_1, \dots, R_k$ is the set of all n such that $[n] = \{0, \dots, n-1\} \models \exists R_1 \dots \exists R_n \psi$. Thus, a spectrum is the class of models of an existential second-order sentence with empty vocabulary.

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A generalized spectrum $\mathscr{K}$ is the class of of finite models of an existential second-order sentence with arbitrary vocabulary.

Fagin's Theorem. $\mathscr{K}$ is in NP if, and only if, $\mathscr{K}$ is a generalized spectrum.

Corollary. (Jones, Selman) $S \subseteq \mathbb{N}$ is a spectrum if, and only if, $S \in$NEXPTIME

Metafinite Spectra

$M_\tau[\mathfrak{R}]$: metafinite structures of vocabulary τ with secondary part $\mathfrak{R}$.

A class $\mathscr{K} \subseteq M_\tau[\mathfrak{R}]$ is a metafinite spectrum if there exists a $\psi \in \mathrm{FO}$ of vocabulary $\sigma \supseteq \tau$ such that $\mathfrak{D} \in \mathscr{K}$ if, and only if, $\mathfrak{D}$ can be expanded to some $\mathfrak{D}^* \in M_\sigma[\mathfrak{R}]$ with $\mathfrak{D}^* \models \psi$. (Note that the secondary part $\mathfrak{R}$ is not expanded.)

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This corresponds to two variants of existential second-order logic for metafinite structures, depending on whether second-order quantifiers range only over relations on the primary part, or also over weight functions.

Both variants of metafinite spectra capture (suitable variants of) NP in certain contexts, but fail to do so in others.

Complexity

Notions of complexity for problems on metafinite structures $\mathfrak{D} = (\mathfrak{A}, \mathfrak{R}, W)$ depend on the model of computation and the cost associated with elements of $\mathfrak{R}$.

For arithmetical structures, our cost measure for $n \in \mathbb{N}$ is the number of bits. But we shall also consider structures with secondary part over $\mathbb{R}$ where $\|r\| = 1$ for all $r \in \mathbb{R}$.

Fix a cost $\|r\| \in \mathbb{N}$ for all $r \in \mathfrak{R}$, and let
$\max \mathfrak{D} := \max\{\|w(\overline{a})\| : w \in W, \overline{a} \in A^k\}$ and $|\mathfrak{D}| := |\mathfrak{A}|$.

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A class $\mathscr{K}$ of metafinite structures has small weights if $\max \mathfrak{D} = |\mathfrak{D}|^{O(1)}$ for all $\mathfrak{D} \in \mathscr{K}$.

Generalizing Fagin's Theorem to Metafinite Structures

First variant of a generalization of Fagin's Theorem to metafinite structures.

Theorem. Let $\mathscr{K} \subseteq M_\tau[\mathfrak{N}]$ be any class of arithmetical structures with small weights. Then $\mathscr{K}$ is a primary metafinite spectrum if, and only if, $\mathscr{K}$ is in NP.

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It is tempting to use unrestricted metafinite spectra instead, corresponding to existential second-order logic with quantification over weight functions.

However these capture a larger class than NP.

Metafinite spectra and Hilbert's 10th Problem

Theorem. On arithmetical structures, metafinite spectra capture the recursively enumerable sets.

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For the converse, let $\mathscr{K} \subseteq M_\tau[\mathfrak{N}]$ be an r.e. set of arithmetical structures. Expand structures $\mathfrak{D} = (\mathfrak{A}, \mathfrak{N}, W) \in M_\tau[\mathfrak{N}]$ by a bijective ranking $r : A \to \{0, \ldots, n-1\}$. There is a first-order definable encoding of ranked structures $(\mathfrak{D}, r)$ in $\mathbb{N}^k$, such that
$\text{code}(\mathscr{K}) := \{\text{code}(\mathfrak{D}, r) : \mathfrak{D} \in \mathscr{K}, r \text{ is a ranking of } \mathfrak{D}\} \subseteq \mathbb{N}^k$ is also r.e.

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By Matijasevich's Theorem, every r.e. set is Diophantine, i.e.

$$\mathrm{code}(\mathscr{K}) = \{\overline{a} \in \mathbb{N}^k : \text{ there exist } b_1 \dots b_m \in \mathbb{N} \text{ with } P(\overline{a}, \overline{b}) = P'(\overline{a}, \overline{b})\}$$

for polynomials $P, P' \in \mathbb{N}[\overline{x}, \overline{y}]$.

Thus code($\mathscr{K}$), and hence also $\mathscr{K}$, is a metafinite spectrum.

The Blum-Shub-Smale model for computation over $\mathbb{R}$

- the input space is $\mathbb{R}^*$: tuples of reals of any finite length
- real numbers are treated as basic entities
- arithmetic operations and tests for zero can be performed in a unit step, for numbers of whatever magnitude or complexity

A BSS machine is basically a RAM over the reals. It abstracts from the complexity of individual reals, approximations, the difficulties of testing whether two represenions of reals denote the same number, and so on.

Certain problem remain intrinsically hard even under such assumptions.

Complexity classes: $\mathrm{P}_{\mathbb{R}}$, $\mathrm{NP}_{\mathbb{R}}$, and so on.

4-FEASIBILITY: Decide whether a given multivariate polynomial of degree 4 has a real root.

Theorem (Blum, Shub, Smale) 4-FEASIBILITY is $\mathrm{NP}_{\mathbb{R}}$-complete.

Describing 4-FEASIBILITY

$\mathbb{R}$-structures: Metafinite structures where the secondary part is $\mathfrak{R} = (\mathbb{R}, +, -, \cdot, /, \leq, (c_r)_{r\in\mathbb{R}})$.

Any polynomial $f(X_1, \dots, X_n) \in \mathbb{R}[\overline{X}]$ of degree 4 can be represented as an $\mathbb{R}$-structure, with primary part $A = (\{0, \dots, n\}, <)$ and a weight function $c : A^4 \to \mathbb{R}$ which describes the monomials $c(i, j, k, \ell)X_iX_jX_kX_\ell$. This gives a homogenous polynomial $g(X_0, X_1, \dots, X_n)$ of degree 4. Setting $X_0 = 1$ gives an arbitrary polynomial of degree ≤ 4 in $X_1, \dots, X_n$.

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To express that f has a real root, existentially quantify over two functions $Z : A \to \mathbb{R}$ and $Y : A^4 \to \mathbb{R}$ such that Z describes the zero, and $Y\overline{x}$ is the partial sum of the monomials up to $\overline{x}$. To do so, state that

- $Y(\overline{0}) = c(\overline{0})$
- for all $\overline{y} = \overline{x} + 1$: $Y(\overline{y}) = Y(\overline{x}) + c(\overline{y})Zy_1 \cdot Zy_2 \cdot Zy_3 \cdot Zy_4$
- $Y(\overline{n}) = 0$.

Descriptive Complexity over $\mathbb{R}$

Theorem (Grädel, Meer). A class of $\mathbb{R}$-structures is a metafinite spectrum if, and only if, it is in $\mathrm{NP}_{\mathbb{R}}$.

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Also the Immerman-Vardi Theorem, saying that fixed-point logic captures polynomial time on ordered finite structures has its analogue for $\mathbb{R}$-structures.

Fixed-point logics and polynomial time

Functional fixed point calculus FFP, based on expressions $\mathbf{fp}[Z\bar{x} \leftarrow F(Z,\bar{x})]$.

Here, Z is a variable for partial functions taking values in $\mathbb{R}$, subject to the

Update rule: $Z \mapsto Z'$ with $Z'(\bar{a}) := \begin{cases} Z(\bar{a}) & \text{if this is defined} \\ F(Z,\bar{a}) & \text{otherwise} \end{cases}$

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As in finite models, these logics are weaker than PTIME if no ranking or linear ordering is available. We do not know whether there is a logic for polynomial time in such cases.

Conclusion

Metafinite model theory seems an adequate approach to deal with hybrid objects, consisting of abstract structures and numbers. It preserves the spirit of finite model theory, that symmetries should be respected.

Most of the methods of finite model theory generalize on some way to metafinite models: Logical descriptions of computation, variants of Ehrenfeucht-Fraïssé games, asymptotic probabilities, and so on …

Disadvantage: Metafinite model theory deals more complicated and hybrid objects, so clearly, the methods and results become more involved and somewhat less elegant. And of course, the main stumbling blocks and unsolved problems of finite model theory survive the transition to metafinite models.