On the Consistency Problem for MODULAR LATTICES and Related Structures

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Geometric Quantum Logic

For fixed $d\in\mathbb{N}$, consider the ortholattices $L(\mathbb{R}^d)$ or $L(\mathbb{C}^d)$ of all linear subspaces of $\mathbb{R}^d$ or $\mathbb{C}^d$, equipped with

$X\wedge Y := X\cap Y, \qquad X\vee Y := X+Y, \qquad \neg X := X^\perp$

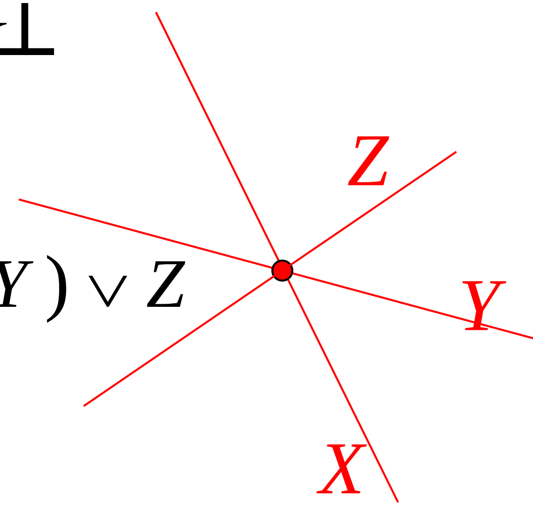
$d=1$: Boolean

$d>1$: still *de Morgan*,

self-dual, neutral $\{0\}=\mathbf{0}$, $\mathbf{1}=\mathbb{R}^d$

–but not distributive; instead **modular law**

$$(X\vee Z)\wedge(Y\vee Z) = ((X\vee Z)\wedge Y)\vee Z$$



–von Neumann'55, Birkhoff'67, Beran'85, Herrmann'72ff

Theorem [Herrmann&Z.10ff]:

$\mathbf{NP}_{\mathbb{R}}$-complete (BSS machine)

Given a term $t(\underline{x})$ in the language of *ortho*lattices, does it admit a *strongly* satisfying assignment $t(\underline{a}) \neq 0$ over $L(\mathbb{R}^2)$? $d\geq 3$

Varieties of Quantum Logic

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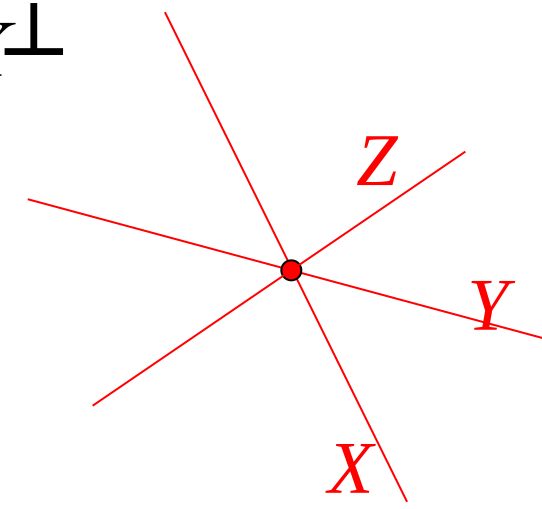

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Theorem [Herrmann&Z.10ff]:

$\mathbf{NP}_{\mathbb{R}}$-hard

$\mathbf{NP}_{\mathbb{R}}$-complete (BSS machine)

Decidable?

$L(\mathbb{R}^d)$ for <u>some</u> $d \neq 0$

Given <u>lattice</u> $\vee\wedge\vee$-term $s(\underline{x})$ and <u>lattice</u> $\wedge\vee\wedge$-term $t(\underline{x})$, is there an assignment $\underline{a}$ over $L(\mathbb{R}^d)$ s.t. $s(\underline{a})=\mathbf{0}$ & $t(\underline{a})=\mathbf{1}$?

$d \geq 3$

Consistency in a Class

Let C denote a class of structures of same signature σ.

Consider a conjunction $\pi(\underline{x})$ of $\boxed{\text{equations}}$ $s_j(\underline{x})=t_j(\underline{x})$ of terms s,t in the language of σ in variables $\underline{x}=(x_1,\ldots,x_n)$.

π is satisfiable over C if there exists $A\in C$ and $\underline{a}\in A$ s.t. $\pi(\underline{a})$.

Call π consistent over C if there is $A\in C$, $\#|A|\geq 1$ and $\underline{a}\in A$ s.t. $\pi(\underline{a})$.

avoid *trivial* cases

- $\Leftrightarrow$ $\pi(\underline{x})$ satisfiable over C \ singleton structures
- $\Leftrightarrow$ $\pi' := \pi(\underline{x})\ \&\ x_0 \neq x_1$ satisfiable over C
- $\Leftrightarrow$ π satisfiable by some assignment $\underline{a}\in A\in C$ *generating* a *non*-singleton subalgebra of A.

if C closed under homom

Theorem [Herrmann&Z.10ff]:

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Consistency as Decision Problem

Let C denote a class of structures of same signature σ.

Consider a conjunction $\pi(\underline{x})$ of equations $s_j(\underline{x})=t_j(\underline{x})$ of terms s,t in the language of σ in variables $\underline{x}=(x_1,\ldots,x_n)$.

π <u>consistent</u> over C if there is $A\in C$, $\#|A|>1$ and $\underline{a}\in A$ s.t. $\pi(\underline{a})$.

		C
Example: b) Given finitely many "relations" $s_j(\underline{x})=e$, is the group G they freely generate trivial?	<u>un</u>decidable [Adyan], [Rabin] 1950ies	all / finitely generated
c) Is G's *profinite completion* trivial?	<u>un</u>decidable [Bridson&Wilton] 2015	finite groups
d) Given integer polynomials in *non*commuting variables, do they have a common root in $L(\mathbb{R}^{d\times d})$ for some $d\neq 0$	<u>un</u>decidable! Decidable?	rings

a) Given <u>lattice</u> $\vee\wedge\vee$-term $s(\underline{x})$ and <u>lattice</u> $\wedge\vee\wedge$-term $t(\underline{x})$, is there an assignment $\underline{a}$ over $L(\mathbb{R}^{+})$ s.t. $s(\underline{a})=\mathbf{0}$ & $t(\underline{a})=\mathbf{1}$?

<u>un</u>decidable!

Reduction from the Profinite Case

Given finitly many equations $s_j(\underline{x})=e$ in the language of groups, is there a $\boxed{\text{nontrivial}}$ finite group satisfying them? <u>un</u>decidable [Bridson&Wilton'15]

Central Tool: Coordinatization

(*von Staudt*, *von Neumann*, Lipshitz'74, Herrmann)

Fix vector space $V = V_1 \vee V_2 \vee V_3 \vee V_4$ with $V_i \wedge V_j = \mathbf{0}$ and isom.s $\varphi_{ij}: V_i \to V_j$ s.t. $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, $V_{ij} := \{\, x - \varphi_{ij}(x) : x \in V_i \,\}$

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i) $V_i \vee V_j = V_i \vee V_{ij}$, $V_i \wedge V_{ij} = \mathbf{0}$, $V_{ik} = (V_i \vee V_k) \wedge (V_{ij} \vee V_{jk})$

ii) For any $A \in \mathrm{GL}(V_1)$, $\Theta(A) := \{\, x - \varphi_{12}(A(x)) : x \in V_1 \,\}$
has $\Theta(A) \wedge V_{1/2} = \mathbf{0}$ and $\Theta(A) \vee V_{1/2} = V_1 \vee V_2$

iii) $U \wedge V_{1/2} = \mathbf{0}$ & $U \vee V_{1/2} = V_1 \vee V_2$ requires $U = \Theta(A)$ for some $A \in \mathrm{GL}(V_1)$

iv) $\Theta(\boldsymbol{B} \circ \boldsymbol{A}) = \Big(\big(\Theta(\boldsymbol{B}) \vee \big(((\Theta(\boldsymbol{A}) \vee V_{23}) \wedge (V_1 \vee V_3)) \vee V_{12}\big) \wedge (V_2 \vee V_3)\big) \wedge (V_1 \vee V_3)\Big) \wedge (V_1 \vee V_2)$

Fact: $V_{ij} \in \mathcal{L}$ mod.lattice s.t. (i) $\Rightarrow$ $\Theta: \mathrm{GL}(V_1) \to \{\text{(iii)}\}$ group isom.

Fact: Any group is isom. to $\mathrm{GL}(V_1)$ for *some* vector space V_1.

Reduction from the Profinite Case

Given finitly many equations $s_j(\underline{x})=e$ in the language of groups, is there a $\boxed{\text{nontrivial}}$ finite group satisfying them?

$\forall g\in G$: $gw=w$ $\Rightarrow$ $w=0$.

Lemma: a) A subgroup G of $\mathrm{GL}(W)$, generated by $g_1,\ldots g_k$, is fixedpoint-free iff $V_{12}\wedge\Theta(g_1)\wedge\ldots\wedge\Theta(g_k)=\mathbf{0}$.
b) For G a *non*-trivial group and V vector space of $\dim(W)\geq|G|$, G has a fixedpoint-free faithful representation in $\mathrm{GL}(W)$.

The trivial group has a representation in a *non*-trivial lattice!

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i) $V_i\vee V_j = V_i\vee V_{ij}$, $V_i\wedge V_{ij} = \mathbf{0}$, $V_{ik} = (V_i\vee V_k)\wedge(V_{ij}\vee V_{jk})$

ii) For any $A\in\mathrm{GL}(V_1)$, $\Theta(A) := \{\ x-\varphi_{12}(A(x)) : x\in V_1\ \}$
has $\Theta(A)\wedge V_{1/2}= \mathbf{0}$ and $\Theta(A)\vee V_{1/2}=V_1\vee V_2$

iii) $U\wedge V_{1/2}=\mathbf{0}$ & $U\vee V_{1/2}=V_1\vee V_2$ requires $U=\Theta(A)$ for some $A\in\mathrm{GL}(V_1)$

iv) $\Theta(\boldsymbol{B}\circ\boldsymbol{A}) = \Big(\big(\Theta(\boldsymbol{B})\vee\big(((\Theta(\boldsymbol{A})\vee V_{23})\wedge(V_1\vee V_3))\vee V_{12}\big)\wedge(V_2\vee V_3)\big)\Big)\wedge\big(V_1\vee V_3\big)\Big)\wedge(V_1\vee V_2)$

Fact: $V_{ij}\in\mathcal{L}$ mod.lattice s.t. (i) $\Rightarrow$ $\Theta:\mathrm{GL}(V_1)\to\{(iii)\}$ group isom.

Fact: Any group is isom. to $\mathrm{GL}(W)$ for *some* vector space W.

Conclusion

There are ortholattice terms t admiting strongly satisfying assignments $\underline{a}$ in $\mathcal{L}(\mathbb{R}^d)$ for some d, but none recursively bounded in $|t|$.

[Hermann'10]: Every weakly satisfiable t is so in some $d \leq O(|t|^2)$

Similarly for int. polynomial systems in noncommuting variables.

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d) Given integer polynomials in *non*commuting variables, do they have a common root in $\mathbb{R}^{d\times d}$	<u>undecidable!</u>	rings for <u>some</u> $d\neq 0$

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<u>undecidable!</u>